Program: RFEM 5, RSTAB 8

Category: Geometrically Linear Analysis, Isotropic Linear Elasticity, Elastic Foundation, Member

Verification Example: 0003 – Cantilever Beam on an Elastic Pasternak Foundation

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Description

A cantilever beam of length L and rectangular cross-section with height h and width b is lying on a Pasternak foundation with stiffness $C_{2,z}$ and loaded by the distributed loading q_z. The elastic Winkler foundation stiffness $C_{1,z}$ is considered zero. Neglecting self-weight, determine the maximum deflection u_z and maximum bending moment M_y of the beam. Calculate these properties for a plate of the same heigth and width as the cantilever, as well.

Material	Isotropic Linear Elastic	Modulus of Elasticity	E	210.000	GPa
		Shear Modulus	G	105.000	GPa
Geometry	Cantilever	Length	L	4.000	m
		Height	h	0.200	m
		Width	b	0.005	m
Member Foundation	Pasternak	Stiffness	$C_{2,z}$	2000.000	kN
Plate Foundation			$C_{v,xz} = \frac{C_{2,z}}{b}$	400000.000	kN/m
Load	Member	Distributed	q_z	1.000	kN/m
	Plate	Distributed	$q = \frac{q_z}{b}$	200.000	kN/m^2

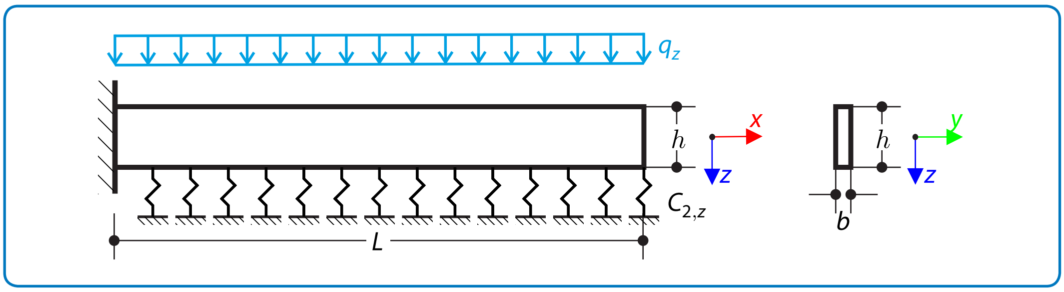

Figure 1: Problem sketch

Analytical Solution

Member Calculation

The governing differential equation of a beam on a Pasternak foundation is given by

$$EI_y \frac{d^4 u_z}{dx^4} - C_{2,z} \frac{d^2 u_z}{dx^2} = q_z \qquad (3-1)$$

where the moment of inertia $I_y = \frac{1}{12}bh^3 = 3.\overline{33} \times 10^{-6}\,\mathrm{m}^4$, E is the Young's modulus of the material and $C_{2,z}$ is the Pasternak foundation stiffness for the beam. Dividing by EI_y, **(3 – 1)** can be rewritten as

$$\frac{\mathrm{d}^4 u_z}{\mathrm{d}x^4} - \underbrace{\frac{C_{2,z}}{EI_y}}_{\alpha} \frac{\mathrm{d}^2 u_z}{\mathrm{d}x^2} = \underbrace{\frac{q_z}{EI_y}}_{A} \tag{3 – 2}$$

where new constants $\alpha = \frac{C_{2,z}}{EI_y}$ and $A = \frac{q_z}{EI_y}$ were defined. The characteristic equation $\lambda^4 - \alpha\lambda^2 = 0$ yields the following fundamental set of solutions of the characteristic equation

$$1,\ x,\ \mathrm{e}^{\sqrt{\alpha}x},\ \mathrm{e}^{-\sqrt{\alpha}x} \tag{3 – 3}$$

A particular solution of **(3 – 2)** is a quadratic polynomial in the form

$$-\frac{Ax^2}{2\alpha} \tag{3 – 4}$$

The solution of **(3 – 2)** is then given by

$$u_z(x) = C_1 + C_2 x + C_3 \mathrm{e}^{\sqrt{\alpha}x} + C_4 \mathrm{e}^{-\sqrt{\alpha}x} - \frac{Ax^2}{2\alpha} \tag{3 – 5}$$

Let us comptue the derivatives of **(3 – 5)**

$$u_z'(x) = C_2 + C_3\sqrt{\alpha}\mathrm{e}^{\sqrt{\alpha}x} - C_4\sqrt{\alpha}\mathrm{e}^{-\sqrt{\alpha}x} - \frac{Ax}{\alpha} \tag{3 – 6}$$

$$u_z''(x) = C_3\alpha\mathrm{e}^{\sqrt{\alpha}x} + C_4\alpha\mathrm{e}^{-\sqrt{\alpha}x} - \frac{A}{\alpha} \tag{3 – 7}$$

$$u_z'''(x) = C_3\alpha^{\frac{3}{2}}\mathrm{e}^{\sqrt{\alpha}x} - C_4\alpha^{\frac{3}{2}}\mathrm{e}^{-\sqrt{\alpha}x} \tag{3 – 8}$$

The constants C_1, C_2, C_3, C_4 are determined by four boundary conditions which are required by the differential equation of the fourth order. These boundary conditions are taken as follows

$$u_z(0) = 0 \tag{3 – 9}$$

$$u_z'(0) = 0 \tag{3 – 10}$$

$$u_z''(L) = 0 \quad \text{(zero moment)} \tag{3 – 11}$$

$$u_z'''(L) - \alpha u_z'(L) = 0 \quad \text{(zero shear force)} \tag{3 – 12}$$

which yields the linear system of equations

$$C_1 + C_3 + C_4 = 0 \tag{3 – 13}$$

$$C_2 + C_3\sqrt{\alpha} - C_4\sqrt{\alpha} = 0 \tag{3 – 14}$$

$$C_3 e^{\sqrt{\alpha}L} + C_4 e^{-\sqrt{\alpha}L} = \frac{A}{\alpha^2} \tag{3 – 15}$$

$$C_2 = \frac{AL}{\alpha} \tag{3 – 16}$$

having the solution

$$C_1 = -\frac{A}{\alpha^2}\left[\frac{1-\sqrt{\alpha}Le^{-\sqrt{\alpha}L}}{\cosh(\sqrt{\alpha}L)} + \sqrt{\alpha}L\right] \tag{3 – 17}$$

$$C_2 = \frac{AL}{\alpha} \tag{3 – 18}$$

$$C_3 = \frac{A}{\alpha^2}\left[\frac{1-\sqrt{\alpha}Le^{-\sqrt{\alpha}L}}{2\cosh(\sqrt{\alpha}L)}\right] \tag{3 – 19}$$

$$C_4 = \frac{A}{\alpha^2}\left[\frac{1-\sqrt{\alpha}Le^{-\sqrt{\alpha}L}}{2\cosh(\sqrt{\alpha}L)} + \sqrt{\alpha}L\right] \tag{3 – 20}$$

The final solution can then be written as

$$u_z(x) = \frac{A}{\alpha}\left(Lx - \frac{x^2}{2}\right) + \frac{A}{\alpha^2}\left[\frac{1-\sqrt{\alpha}Le^{-\sqrt{\alpha}L}}{\cosh\sqrt{\alpha}L}(\cosh\sqrt{\alpha}x - 1) + \sqrt{\alpha}L\left(e^{-\sqrt{\alpha}x} - 1\right)\right] \tag{3 – 21}$$

where $\cosh(x) = \frac{e^x + e^{-x}}{2}$. Hence, from equation **(3 – 21)** the following maximum deflection can be deduced

$$u_{z,\max} = u_z(L) = 2.991 \text{ mm} \tag{3 – 22}$$

while the maximum of the bending moment M_y evaluates to

$$M_{y,\max} = M_y(0) = -EI_y\frac{d^2u_z}{dx^2}(0) = \frac{A}{\alpha}\left[\frac{1-\sqrt{\alpha}Le^{-\sqrt{\alpha}L}}{\cosh(\sqrt{\alpha}L)} + \sqrt{\alpha}L - 1\right] = -2.017 \text{ kNm} \tag{3 – 23}$$

Plate Calculation

The theory is identical, the parameter describing the Pasternak foundation for plates $C_{2,z}$ equals to

$$C_{v,xz} = \frac{C_{2,z}}{b} = 400000 \text{ kN/m} \tag{3 – 24}$$

Note that the Poisson ratio is zero in order to approximate the member solution exactly.

RFEM 5 and RSTAB 8 Settings

- Modeled in version RFEM 5.16.01 and RSTAB 8.16.01
- The element size is $l_{FE} = 0.100$ m
- Geometrically linear analysis is considered
- Isotropic linear elastic material model is used
- The Kirchhoff plate theory is used
- Shear stiffness of members is deactivated

Results

Structure File	Entity	Program
0003.01	Member	RFEM 5
0003.02	Member	RSTAB 8
0003.03	Plate	RFEM 5

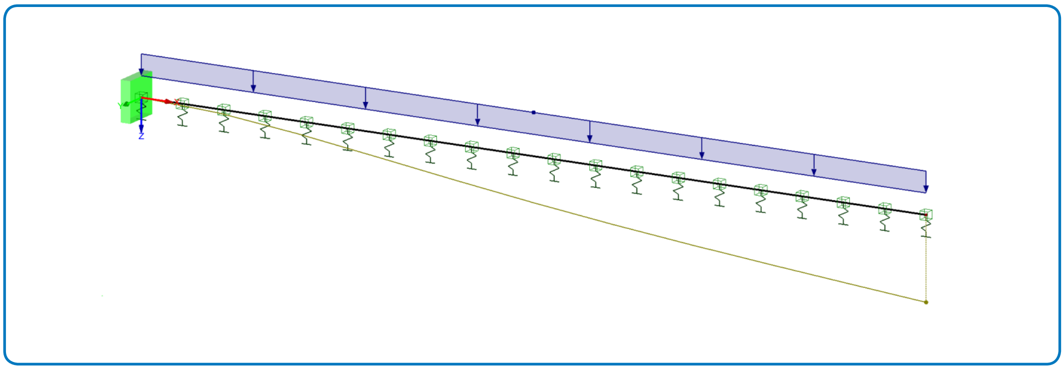

Figure 2: RFEM 5 Model

As can be seen from the following comparison, excellent agreement between the analytical solutions and the numerical outputs was achieved.

Analytical Solution	RFEM 5 (Member)		RSTAB 8 (Member)		RFEM 5 (Plate)	
$u_{z,max}$ [mm]	$u_{z,max}$ [mm]	Ratio [-]	$u_{z,max}$ [mm]	Ratio [-]	$u_{z,max}$ [mm]	Ratio [-]
2.991	2.991	1.000	2.991	1.000	3.005	1.005

Analytical Solution	RFEM 5 (Member)		RSTAB 8 (Member)		RFEM 5 (Plate)	
$M_{y,max}$ [kNm]	$M_{y,max}$ [kNm]	Ratio [-]	$M_{y,max}$ [kNm]	Ratio [-]	$m_{x,max} \times b$ [kNm]	Ratio [-]
−2.017	−2.017	1.000	−2.013	0.998	−1.999	0.991