0020 – Plastic Bending with Different Plastic Strengths

Description

A cantilever is fully fixed on the left end ($x = 0$) and subjected to a bending moment M on the right end according to the **Figure 1**. The material has different plastic strengths in tension and compression. The problem is described by the following set of parameters.

Material	Elastic-Plastic	Modulus of Elasticity	E	210000.000	MPa
		Poisson's Ratio	ν	0.000	–
		Shear Modulus	G	105000.000	MPa
		Tensile Plastic Strength	f_t	200.000	MPa
		Compressive Plastic Strength	f_c	280.000	MPa
Geometry	Cantilever	Length	L	2.000	m
		Width	w	0.005	m
		Thickness	t	0.005	m
Load		Bending Moment	M	6.000	Nm

Small deformations are considered and the self-weight is neglected in this example. Determine the maximum deflection $u_{z,\max}$.

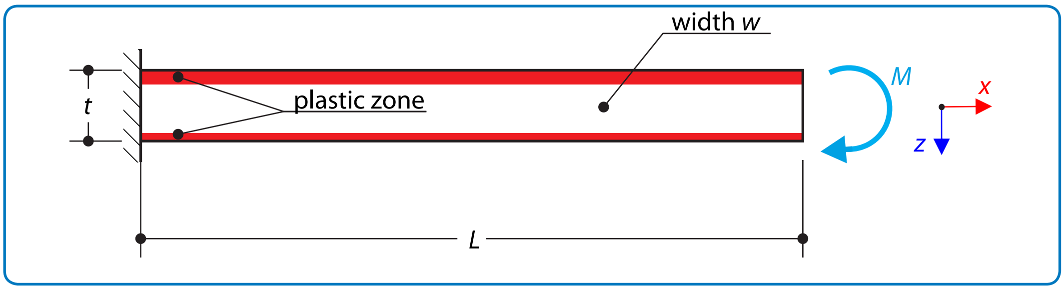

Figure 1: Problem sketch

Analytical Solution

Linear Analysis

Considering linear analysis (only elasticity) the maximum deflection of the structure can be calculated as follows:

$$u_{z,\max} = \frac{ML^2}{2EI_y} = 1.097 \text{ m} \qquad (20-1)$$

Nonlinear Analysis

The cantilever is loaded by the bending moment M. Due to the different plastic strength in the tension and compression the neutral axis is not necessary coincident with the axis of the symmetry according to the **Figure 2**. The parameter z_0 is introduced and it is defined so that $\sigma_x(x, z_0) = 0$, note that it changes during loading as well as parameters z_t and z_c. The bending stress is defined by the following formula

$$\sigma_x = -\kappa E(z - z_0(x)) \qquad (20-2)$$

where $\kappa(x)$ is the curvature defined as $\kappa(x) = \mathrm{d}^2 u_z / \mathrm{d}x^2$ [1].

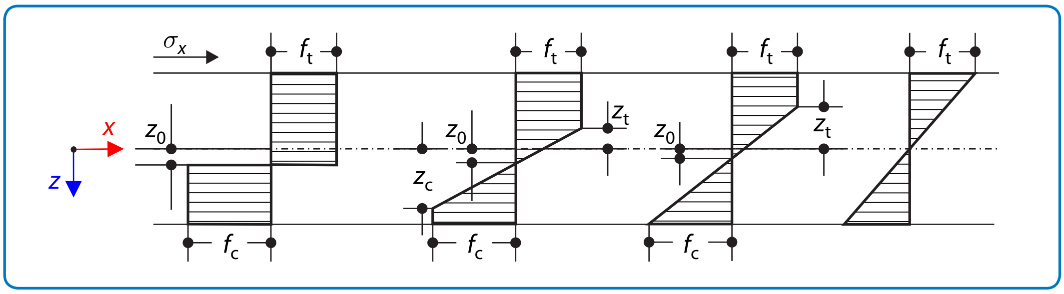

Figure 2: Bending stress distribution

The quantities of the bending moment M are discussed at first. The moment M_{et} when the first yield occurs in the tensile part, the moment M_{ec} when the first yield occurs in the compressed part and the ultimate moment M_p when the structure becomes plastic hinge are calculated as follows, assuming $f_t < f_c$

$$M_{et} = 2\int_{-t/2}^{0} \sigma(z) zw \, \mathrm{d}z = 2\int_{-t/2}^{0} -\frac{2f_t}{t} z^2 w \, \mathrm{d}z = -\frac{f_t w t^2}{6} = -4.1\bar{6} \text{ Nm} \qquad (20-3)$$

$$M_{ec} = \int_{-t/2}^{-z_t} f_t zw \, \mathrm{d}z + \int_{-z_t}^{t/2} -\kappa E(z - z_0) zw \, \mathrm{d}z = -5.614 \text{ Nm} \qquad (20-4)$$

$$M_p = \int_{-t/2}^{z_0} f_t zw \, \mathrm{d}z + \int_{z_0}^{t/2} -f_c zw \, \mathrm{d}z = -7.292 \text{ Nm} \qquad (20-5)$$

where the parameters z_t and z_0 are obtained for each stress state from the equality of the curvature κ and from the equilibrium of the axial forces N in the cross-section **(20 – 6)**.

$$N = \int_A \sigma(z) \, \mathrm{d}A = 0 \qquad (20-6)$$

It is obvious that for the given loading moment M the cantilever is in the elastic-plastic state and both top and bottom edge are in the plastic state. To obtain the maximum deflection $u_{z,\max}$ the curvature κ has to be solved. The elastic-plastic moment M_{ep} (internal force) has to equal to the bending moment M (external force)

$$M_{\mathrm{ep}} = \int_{-t/2}^{-z_{\mathrm{t}}} f_{\mathrm{t}} z w \,\mathrm{d}z + \int_{-z_{\mathrm{t}}}^{z_{\mathrm{c}}} -\kappa E(z - z_0) z w \,\mathrm{d}z + \int_{z_{\mathrm{c}}}^{t/2} -f_{\mathrm{c}} z w \,\mathrm{d}z = M \tag{20 – 7}$$

because of the unknown parameters z_{t}, z_{c} and z_0 it is necessary to write further equations. The stresses in the interface between the elastic and plastic zones are defined as follows.

$$f_{\mathrm{t}} = -\kappa E(-z_{\mathrm{t}} - z_0) \tag{20 – 8}$$

$$-f_{\mathrm{c}} = -\kappa E(z_{\mathrm{c}} - z_0) \tag{20 – 9}$$

The last condition is defined by the equilibrium of the axial forces.

$$N = \int_{-t/2}^{-z_{\mathrm{t}}} f_{\mathrm{t}} w \,\mathrm{d}z + \int_{-z_{\mathrm{t}}}^{z_{\mathrm{c}}} -\kappa E(z - z_0) w \,\mathrm{d}z + \int_{z_{\mathrm{c}}}^{t/2} -f_{\mathrm{c}} w \,\mathrm{d}z = 0 \tag{20 – 10}$$

Solving equations **(20 – 7)**, **(20 – 8)**, **(20 – 9)** and **(20 – 10)** numerically, the curvature κ results in

$$\kappa = 0.636 \text{ m}^{-1} \tag{20 – 11}$$

The maximum deflection $u_{z,\max}$ can be then calculated as

$$u_{z,\max} = \int_{0}^{L} \kappa(L - x) \,\mathrm{d}x = 1.272 \text{ m} \tag{20 – 12}$$

RFEM 5 Settings

- Modeled in RFEM 5.16.01
- The element size is $l_{\mathrm{FE}} = 0.020$ m
- Geometrically linear analysis is considered
- The number of increments is 5
- Shear stiffness of the members is neglected

Results

Structure File	Entity	Material model	Hypothesis
0020.01	Plate	Orthotropic Plastic 2D	Tsai-Wu
0020.02	Member	Isotropic Nonlinear Elastic 1D	-
0020.03	Plate	Nonlinear Elastic 2D/3D	Mohr-Coulomb
0020.04	Plate	Nonlinear Elastic 2D/3D	Drucker-Prager
0020.05	Plate	Isotropic Plastic 2D/3D	Mohr-Coulomb
0020.06	Plate	Isotropic Plastic 2D/3D	Drucker-Prager
0020.07	Solid	Nonlinear Elastic 2D/3D	Mohr-Coulomb
0020.08	Solid	Nonlinear Elastic 2D/3D	Drucker-Prager
0020.09	Solid	Isotropic Plastic 2D/3D	Mohr-Coulomb
0020.10	Solid	Isotropic Plastic 2D/3D	Drucker-Prager

Model	Analytical Solution	RFEM 5	
	$u_{z,max}$ [m]	$u_{z,max}$ [m]	Ratio [-]
Orthotropic Plastic 2D	1.272	1.277	1.004
Isotropic Nonlinear Elastic 1D		1.271	0.999
Nonlinear Elastic 2D/3D,Mohr-Coulomb, Plate		1.284	1.009
Nonlinear Elastic 2D/3D,Drucker-Prager, Plate		1.284	1.009
Isotropic Plastic 2D/3D,Mohr-Coulomb, Plate		1.284	1.009
Isotropic Plastic 2D/3D,Drucker-Prager, Plate		1.272	1.000
Nonlinear Elastic 2D/3D,Mohr-Coulomb, Solid		1.310	1.030
Nonlinear Elastic 2D/3D,Drucker-Prager, Solid		1.312	1.031
Isotropic Plastic 2D/3D,Mohr-Coulomb, Solid		1.293	1.017
Isotropic Plastic 2D/3D,Drucker-Prager, Solid		1.283	1.009

References

[1] LUBLINER, J. *Plasticity theory*. Berkeley: University of California, 1990.