# 0086 – Curved Beam with Out-of-Plane Loading

## Description

A quarter-circle beam with rectangular cross-section $w \times h$ is loaded by means of an out-of-plane force $F$ according to **Figure 1**. While neglecting self-weight, determine the total deflection $u_z$ of the curved beam.

| | | | | |
|---|---|---|---|---|
| Material | Modulus of Elasticity | $E$ | 210000.0 | MPa |
| | Poisson's Ratio | $\nu$ | 0.296 | – |
| Geometry | Radius | $r$ | 1.000 | m |
| | Cross-section Width | $w$ | 25.000 | mm |
| | Cross-section Height | $h$ | 50.000 | mm |
| Load | Force | $F$ | 1.000 | kN |

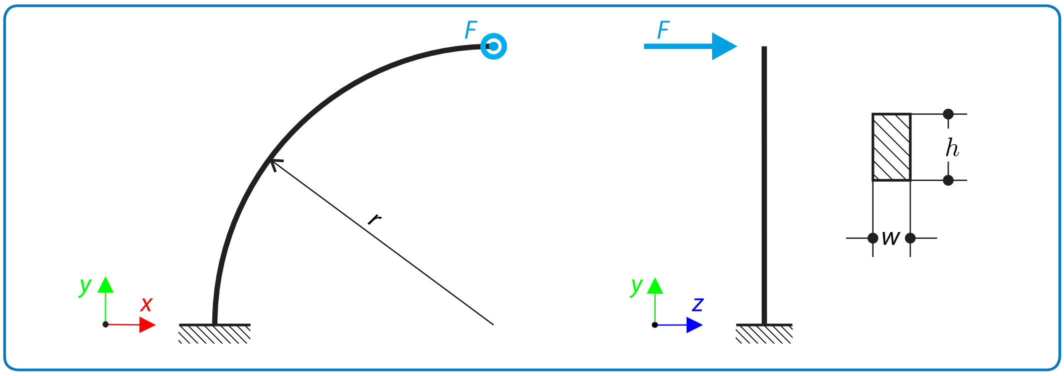


**Figure 1:** Problem Sketch

## Analytical Solution

The curved beam is loaded by a bending moment $M_b$, torsional moment $M_t$ and by a transverse force $T$. Considering the scheme in **Figure 2**, these loads at an arbitrary section are equal to

$$M_b = Fa = Fr\sin\varphi, \tag{86 – 1}$$

$$M_t = Fb = Fr(1-\cos\varphi), \tag{86 – 2}$$

$$T = F. \tag{86 – 3}$$

The deflection of the structure can be then determined according to Castigliano's second theorem

$$u_z = \frac{\mathrm{d}U}{\mathrm{d}F} = \frac{\mathrm{d}(U_b + U_t + U_s)}{\mathrm{d}F}, \tag{86 – 4}$$

where the total strain energy $U$ is composed of the bending strain energy $U_b$, torsional strain energy $U_t$ and shear strain energy $U_s$.

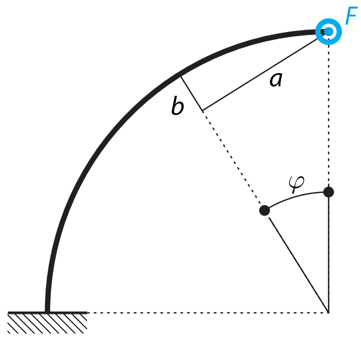


**Figure 2:** Scheme

The strain energy is calculated integrating along the length of the curved beam $L$. Considering polar coordinates, the infinitesimal length of the arc is defined as $\mathrm{d}s = r\mathrm{d}\varphi$.

$$U_b = \int_L \frac{M_b^2(s)}{2EI_y}\mathrm{d}s = \int_0^{\pi/2} \frac{M_b^2(\varphi)}{2EI_y} r\mathrm{d}\varphi \tag{86 – 5}$$

$$U_t = \int_L \frac{M_t^2(s)}{2GJ}\mathrm{d}s = \int_0^{\pi/2} \frac{M_t^2(\varphi)}{2GJ} r\mathrm{d}\varphi \tag{86 – 6}$$

$$U_s = \frac{6}{5}\int_L \frac{T^2}{2GA}\mathrm{d}s = \frac{6}{5}\int_0^{\pi/2} \frac{T^2}{2GA} r\mathrm{d}\varphi \tag{86 – 7}$$

The second moment of the area $I_y$ for the rectangular cross-section is defined as $I_y = \frac{1}{12}wh^3$, the torsional constant $J$ is defined as $J = 0.229hw^3$ because of the particular $h/w$ ratio according to [1] and the cross-section area is equal to $A = wh$. The total deflection of the curved beam $u_z$ is then equal to

$$u_z = \frac{3\pi Fr^3}{Ewh^3} + \frac{1.555Fr^3}{Ghw^3} + \frac{3\pi Fr}{5Ghw} \approx 38.960 \text{ mm}. \tag{86 – 8}$$

## RFEM 5 Settings

- Modeled in RFEM 5.12.02
- Element size is $l_{\mathrm{FE}} = 0.010$ m
- The number of increments is 10
- Isotropic linear elastic material is used
- Mindlin plate bending theory is used

## Results

| Structure File | Entity | Orientation |
|---|---|---|
| 0086.01 | Member | – |
| 0086.02 | Plate | Horizontal |
| 0086.03 | Plate | Vertical |
| 0086.04 | Solid | – |

| Entity | Theory | RFEM 5 | |
|---|---|---|---|
| | $u_z$ [mm] | $u_z$ [mm] | Ratio [-] |
| Member | 38.960 | 38.974 | 1.000 |
| Plate, horizontal | | 38.642 | 0.992 |
| Plate, vertical | | 38.117 | 0.978 |
| Solid | | 38.398 | 0.986 |

## References

[1] https://www.colorado.edu/engineering/CAS/courses.d/Structures.d/, *Introduction to aerospace structures*