0107 – Natural Vibrations of Rectangular Membrane

Description

A rectangular membrane is tensioned by a line force N according to **Figure 1**. Determine the natural frequencies of the given membrane. The problem is described by the following parameters.

Material	Steel	Modulus of Elasticity	E	210000.0	MPa
		Poisson's Ratio	ν	0.300	–
		Density	ρ	7850.000	kgm^{-3}
Geometry		Width	a	1.000	m
		Length	b	1.500	m
		Thickness	h	0.001	m
Load		Tension Force	N	10000.000	Nm^{-1}

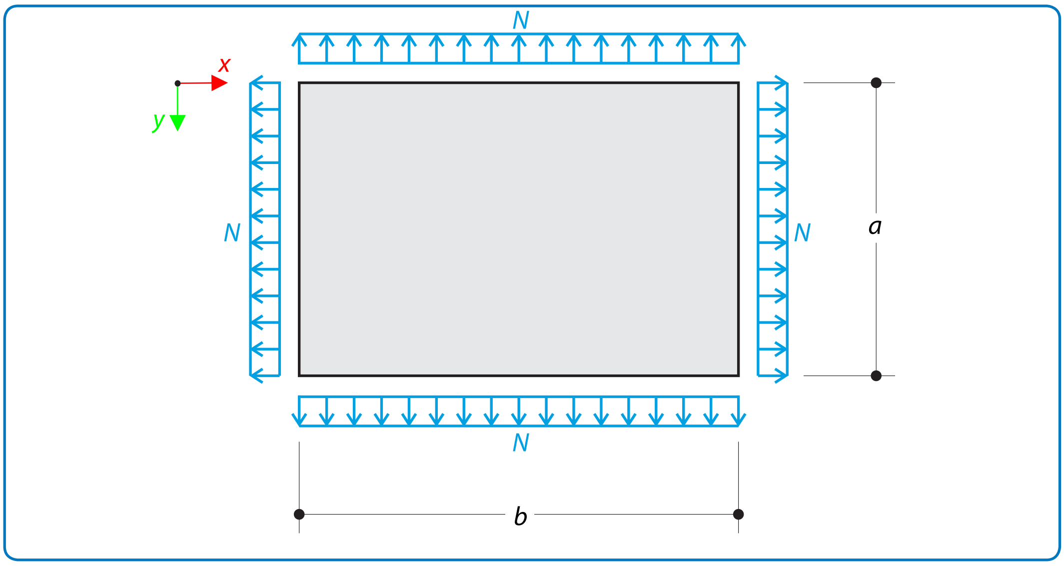

Figure 1: Problem Sketch

Analytical Solution

Free vibrations of a rectangular membrane are described by the wave equation in the following form

$$\frac{\partial^2 u_z}{\partial x^2}(x,y,t) + \frac{\partial^2 u_z}{\partial y^2}(x,y,t) - \frac{1}{c^2}\frac{\partial^2 u_z}{\partial t^2}(x,y,t) = 0 \qquad (107-1)$$

The speed of the wave propagation c is given by the density of the membrane ρ, the membrane thickness h and the tension force N

$$c = \sqrt{\frac{N}{\rho h}} \tag{107 – 2}$$

The solution is sought for through separation of variables

$$u_z(x, y, t) = X(x)Y(y)T(t) \tag{107 – 3}$$

Hence, substituting into **(107 – 1)** yields[1]

$$\frac{\ddot{T}}{T} = -\Omega^2 = c^2 \left(\frac{X''}{X} + \frac{Y''}{Y} \right) \tag{107 – 4}$$

The left-hand side depends on time t, while the right-hand side only on the spatial coordinates x and y. Thus, both sides have to be equal to a constant, which can be shown to be negative, $-\Omega^2$, for some $\Omega > 0$.

The first part of **(107 – 4)**

$$\ddot{T} + \Omega^2 T = 0 \tag{107 – 5}$$

yields a solution in the following form

$$T(t) = A \sin(\Omega t) + B \cos(\Omega t) \tag{107 – 6}$$

where A, B depend on the initial conditions. It follows from the second part of **(107 – 4)**

$$\frac{X''}{X} + \frac{Y''}{Y} = -\frac{\Omega^2}{c^2} \tag{107 – 7}$$

that the left-hand side has to be the sum of two negative constants $-\alpha^2$ and $-\beta^2$, as the two terms contain variables that are independent from each other, more precisely,

$$-\alpha^2 - \beta^2 = -\frac{\Omega^2}{c^2} \tag{107 – 8}$$

Thus, the following two equations have to be solved

$$X'' + \alpha^2 X = 0 \tag{107 – 9}$$

$$Y'' + \beta^2 Y = 0 \tag{107 – 10}$$

[1] The dashed notation indicates the derivative with respect to the space coordinate $X'' = \frac{d^2X(x)}{dx^2}$, while the dotted notation with respect to time t.

The solutions are analogous to the time-variable case

$$X(x) = C\sin(\alpha x) + D\cos(\alpha x) \tag{107 – 11}$$

$$Y(y) = E\sin(\beta y) + F\cos(\beta y) \tag{107 – 12}$$

and the constants C, D, E, F depend on the boundary conditions, which, in this case, state that the deflection on the whole boundary is equal to zero

$$X(0) = 0 \tag{107 – 13}$$

$$X(a) = 0 \tag{107 – 14}$$

$$Y(0) = 0 \tag{107 – 15}$$

$$Y(b) = 0 \tag{107 – 16}$$

Using these boundary conditions, the constants D and F have to be zero. The other two boundary conditions yield

$$C\sin(\alpha a) = 0 \tag{107 – 17}$$

$$E\sin(\beta b) = 0 \tag{107 – 18}$$

which means that α, β are the roots of the sine function, more precisely

$$\alpha_m = \frac{m\pi}{a}, \qquad m = 1, 2, 3, \ldots \tag{107 – 19}$$

$$\beta_n = \frac{n\pi}{b}, \qquad n = 1, 2, 3, \ldots \tag{107 – 20}$$

Considering that $\Omega_{mn} = 2\pi f_{mn}$, when these solutions are substituted into **(107 – 8)** the natural frequencies of the rectangular membrane can be calculated according to the following equation

$$f_{mn} = \frac{c}{2}\sqrt{\frac{m^2}{a^2} + \frac{n^2}{b^2}} \tag{107 – 21}$$

For the first six natural frequencies, for $mn = 11, 12, 21, 13, 22, 23$, of the given membrane see the result table below.

RFEM 5 Settings

- Modeled in RFEM 5.16.01
- The element size is $l_{\mathrm{FE}} = 0.030$ m
- Isotropic linear elastic material model is used

Results

Structure Files	Program
0107.01	RFEM 5 – RF-DYNAM Pro

Quantity	Analytical Solution	RFEM 5	Ratio
f_1 [Hz]	21.448	21.441	1.000
f_2 [Hz]	29.743	29.726	0.999
f_3 [Hz]	37.622	37.573	0.999
f_4 [Hz]	39.904	39.854	0.999
f_5 [Hz]	42.896	42.844	0.999
f_6 [Hz]	50.475	50.401	0.999

Figure 2 shows the first six natural shapes of the investigated membrane.

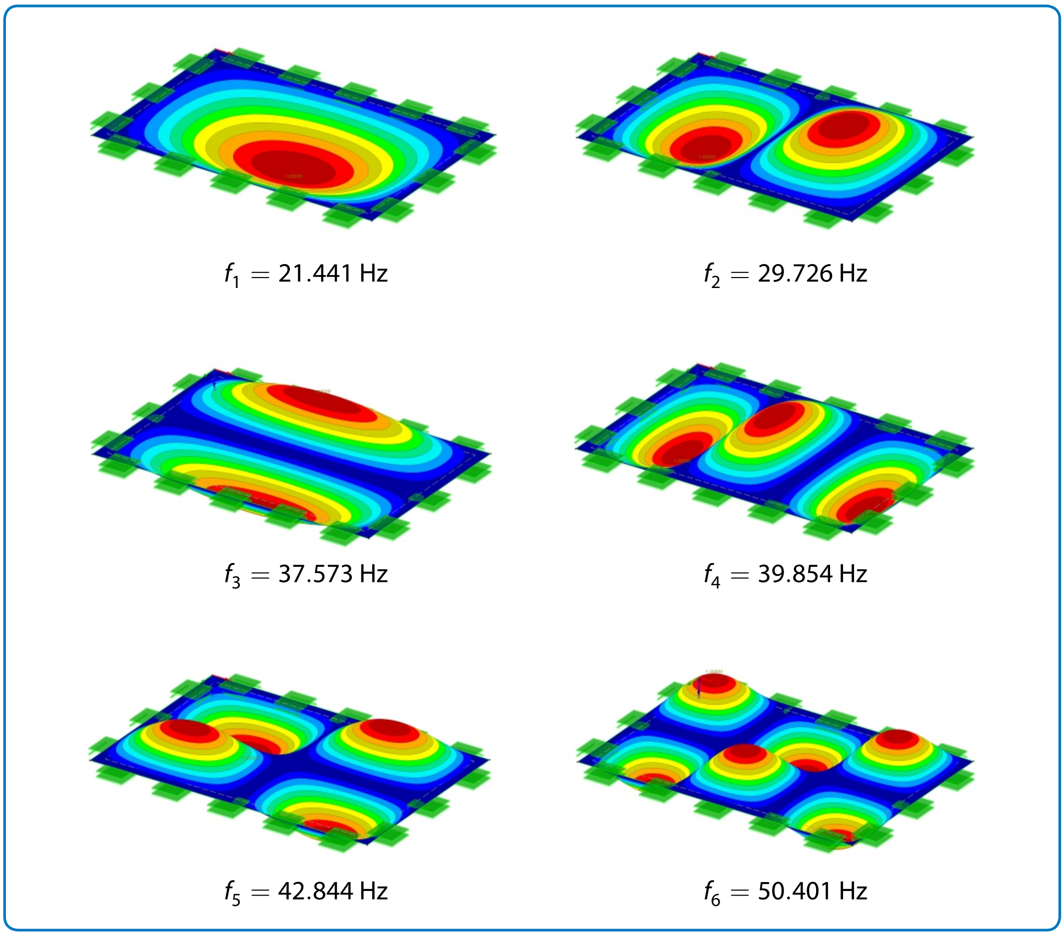

Figure 2: First six natural shapes of the membrane in RFEM 5