0109 – Natural Vibrations of Circular Plate

Description

A circular steel plate of radius a and thickness h is clamped around its circumference $r = a$ according to **Figure 1**. Determine the natural frequencies of the circular plate. The problem is described by the following parameters.

Material	Steel	Modulus of Elasticity	E	210000.0	MPa
		Poisson's Ratio	ν	0.300	–
		Density	ρ	7850.000	kgm^{-3}
Geometry		Radius	a	0.500	m
		Thickness	h	0.001	m

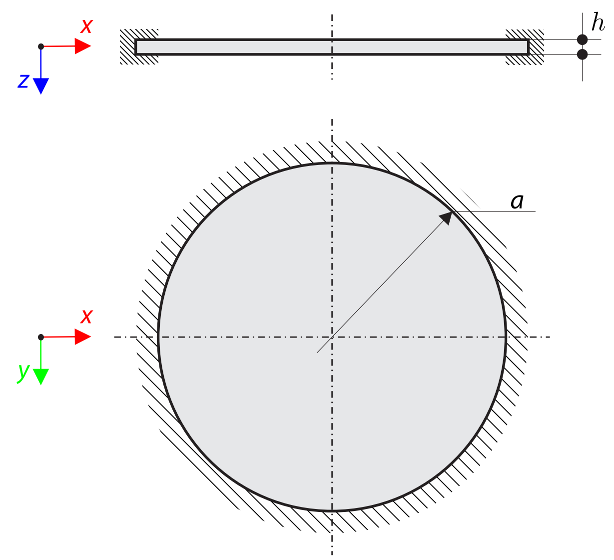

Figure 1: Problem Sketch

Analytical Solution

Free vibrations of a circular plate can be described by the wave equation in polar coordinates r, φ

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \varphi^2}\right)^2 u_z + \frac{1}{c^2}\frac{\partial^2}{\partial t^2} u_z = 0 \qquad (109-1)$$

where $u_z = u_z(r, \varphi, t)$ is the deflection in transversal direction. The speed of the wave propagation c is given by the density of the plate density ρ, plate thickness h and the plate modulus D

$$c = \sqrt{\frac{D}{\rho h}}, \qquad D = \frac{Eh^3}{12(1-\nu^2)} \tag{109 – 2}$$

The solution is sought for in following form

$$u_z(r,\varphi,t) = W(r,\varphi)e^{i\Omega t} \tag{109 – 3}$$

After substitution into **(109 – 1)**, the following equation is obtained

$$\Delta^2 W - \frac{\Omega^2}{c^2} W = 0 \tag{109 – 4}$$

where Δ is the Laplace operator in polar coordinates, more precisely

$$\Delta = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \varphi^2} \tag{109 – 5}$$

hence, **(109 – 4)** can be rewritten as

$$\left(\Delta + \frac{\Omega}{c}\right)\left(\Delta - \frac{\Omega}{c}\right) W = 0 \tag{109 – 6}$$

Thus two independent equations have to be solved further

$$\left(\Delta + \frac{\Omega}{c}\right) = 0, \qquad \left(\Delta - \frac{\Omega}{c}\right) = 0 \tag{109 – 7}$$

Assuming that the variables of $W(r,\varphi)$ are separated, i.e., $W = R(r)F(\varphi)$, **(109 – 7)** yields[1]

$$r^2\frac{R''}{R} + r\frac{R'}{R} \pm r^2\frac{\Omega}{c} = m^2 = -\frac{F''}{F} \tag{109 – 8}$$

The left-hand side of the equation depends on the radius r, and the right-hand side depends on the angle φ, which are independent variables. Thus, both sides have to be equal to a constant m^2.

The angle-dependent equation $F'' + m^2 F = 0$ admits the solution

$$F(\varphi) = A\sin(m\varphi) + B\cos(m\varphi) \tag{109 – 9}$$

where the constants A, B are defined by the boundary conditions.

On the other hand, the radius-dependent equations can be rewritten into the Bessel and modified Bessel differential equations, more precisely

[1] The dashed notation indicates the derivative with respect to the appropriate spatial coordinate, e.g., $R'' = \frac{\mathrm{d}^2 R}{\mathrm{d}r^2}(r)$.

$$r^2 R'' + rR' \pm \left(r^2 \frac{\Omega}{c} - m^2 \right) R = 0 \qquad (109-10)$$

The solution of the Bessel equations take the form

$$R(r) = CJ_m \left(\sqrt{\frac{\Omega_{mn}}{c}} r \right) + DY_m \left(\sqrt{\frac{\Omega_{mn}}{c}} r \right), \qquad n = 1, 2, 3, \ldots \qquad (109-11)$$

$$R(r) = EI_m \left(\sqrt{\frac{\Omega_{mn}}{c}} r \right) + FK_m \left(\sqrt{\frac{\Omega_{mn}}{c}} r \right), \qquad n = 1, 2, 3, \ldots \qquad (109-12)$$

where J_m and Y_m are Bessel functions and I_m and K_m are modified Bessel functions.

For the circular plate, Bessel functions Y_m and K_m become unbounded as $r \to 0$, therefore the constants D, F have to equal zero. The general solution is then rewritten in the form of linear combination with constants C_1, C_2 as

$$R(r) = C_1 J_m \left(\sqrt{\frac{\Omega_{mn}}{c}} r \right) + C_2 I_m \left(\sqrt{\frac{\Omega_{mn}}{c}} r \right) \qquad (109-13)$$

For the clamped plate, the following boundary conditions are prescribed

$$R(a) = C_1 J_m \left(\sqrt{\frac{\Omega_{mn}}{c}} a \right) + C_2 I_m \left(\sqrt{\frac{\Omega_{mn}}{c}} a \right) = 0 \qquad (109-14)$$

$$R'(a) = C_1 J'_m \left(\sqrt{\frac{\Omega_{mn}}{c}} a \right) + C_2 I'_m \left(\sqrt{\frac{\Omega_{mn}}{c}} a \right) = 0 \qquad (109-15)$$

where the dash denotes the derivative of the appropriate Bessel function. These derivatives can be replaced according to general formulas for Bessel functions

$$J'_m(x) = -J_{m+1}(x) + \frac{mJ_m(x)}{x} \qquad (109-16)$$

$$I'_m(x) = I_{m+1}(x) + \frac{mI_m(x)}{x} \qquad (109-17)$$

$$I_m \left(\sqrt{\frac{\Omega_{mn}}{c}} a \right) J_{m+1} \left(\sqrt{\frac{\Omega_{mn}}{c}} a \right) + J_m \left(\sqrt{\frac{\Omega_{mn}}{c}} a \right) I_{m+1} \left(\sqrt{\frac{\Omega_{mn}}{c}} a \right) = 0 \qquad (109-18)$$

From the roots of this equation the set of constants Ω_{mn} is obtained. Considering $\Omega_{mn} = 2\pi f_{mn}$, the set of natural frequencies f_{mn} of the clamped circular plate can be calculated. The first six natural frequencies can be found in the result table.

RFEM 5 Settings

- Modeled in RFEM 5.07.05
- The element size is $l_{FE} = 0.010$ m
- For entity type Solid, layered mesh is used with 4 layers
- Isotropic linear elastic material model is used

Results

Structure Files	Program	Entity
0109.01	RF-DYNAM Pro	Plate
0109.02	RF-DYNAM Pro	Solid

Frequency	Analytical Solution	Plate		Solid	
		RF-DYNAM Pro	Ratio	RF-DYNAM Pro	Ratio
f_1 [Hz]	10.179	10.179	1.000	10.253	1.007
f_2 [Hz]	21.184	21.184	1.000	21.343	1.008
f_3 [Hz]	34.752	34.751	1.000	35.020	1.008
f_4 [Hz]	39.629	39.624	1.000	39.940	1.008
f_5 [Hz]	50.847	50.844	1.000	51.251	1.008
f_6 [Hz]	60.611	60.604	1.000	61.118	1.008

Following **Figure 2** shows the first six natural shapes of the investigated plate.

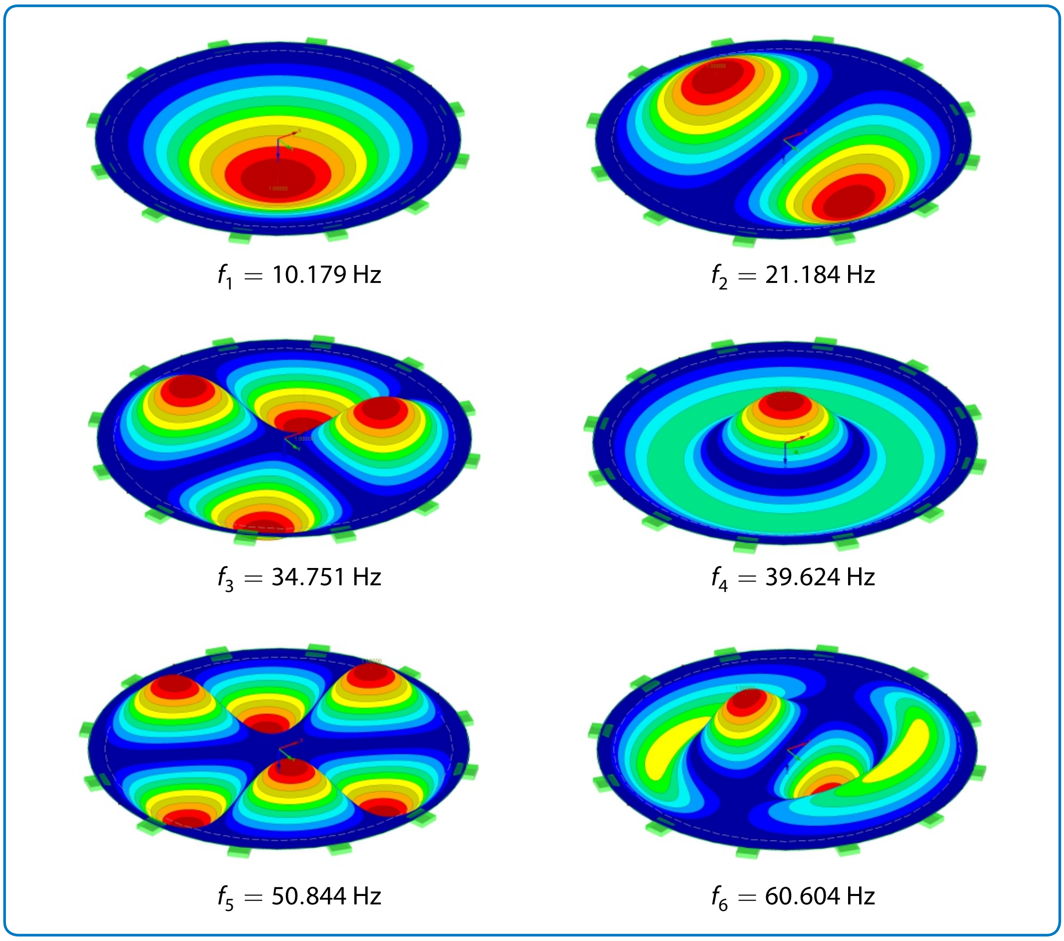

Figure 2: First six natural shapes of the plate in RFEM 5