Program: RFEM 5, RSTAB 8, RF-STABILITY, RSBUCK

Category: Geometrically Linear Analysis, Second-Order Analysis, Isotropic Linear Elasticity, Stability, Member, Plate

Verification Example: 0004 – Stability of a Beam Subjected to the Combined Loading

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Description

A steel beam with a square cross-section is subjected to the axial force F and distributed loading q. Neglecting its self-weight, determine the maximum deflection of the structure $u_{z,\max}$ assuming the Geometrically linear analysis or the second order analysis and the critical load scaling factor b by the linear stability analysis.

Material	Steel	Modulus of Elasticity	E	200.00	GPa
		Poisson's Ratio	ν	0.300	–
Geometry	Beam	Height Width	d	0.010	m
		Length	L	1.000	m
Load		Distributed	q	100.000	N/m
		Force	F	166.700	N

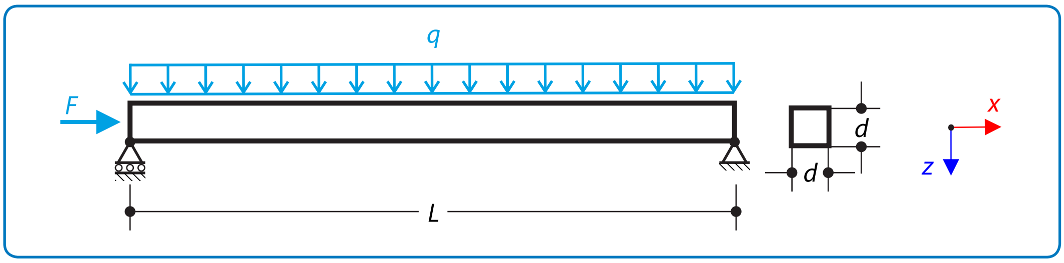

Figure 1: Problem sketch

Analytical Solution

Geometrically Linear Analysis

While considering only the small deformation theory, the beam deflection in the z-direction is not affected by the axial force and can be obtained by the double integration of the following Euler-Bernoulli equation:

$$-EI\frac{\mathrm{d}^2 u_z(x)}{\mathrm{d}x^2} = M \qquad (4-1)$$

where M is the moment on the beam at the distance x from the left end:

$$M = \frac{qL}{2}x - \frac{q}{2}x^2 \qquad (4-2)$$

Double integration gives:

$$-EIu_z(x) = \frac{qL}{12}x^3 - \frac{q}{24}x^4 + Ax + B \tag{4 – 3}$$

Coefficients A and B can be obtained using boundary conditions at the ends of the beam:

$$u_z(0) = 0 \rightarrow B = 0 \tag{4 – 4}$$

$$u_z(L) = 0 \rightarrow A = -\frac{qL^3}{24} \tag{4 – 5}$$

Evaluation of the equation **(4 – 3)** at $x = \frac{L}{2}$ gives the maximum deflection of the beam:

$$u_z\left(\frac{L}{2}\right) = \frac{5}{384}\frac{qL^4}{EI} = 7.813 \text{ mm} \tag{4 – 6}$$

where $I = \frac{h^4}{12}$ is the second moment of the area.

Second-Order Analysis

The second order theory accounts for the effect of the axial force F, which has to be implemented into the expression for the moment on the beam. The equation **(4 – 1)** then gets the following form:

$$-EI\frac{d^2u_z(x)}{dx^2} = \frac{qL}{2}x - \frac{q}{2}x^2 + Fu_z(x) \tag{4 – 7}$$

The solution of the equation **(4 – 7)** can be generally expressed as:

$$u_z(x) = C_1 \sin(\alpha x) + C_2 \cos(\alpha x) + u_{z,\mathrm{p}}(x) \tag{4 – 8}$$

where $\alpha = \sqrt{\frac{F}{EI}}$ and $u_{z,\mathrm{p}}(x)$ is the particular solution of the equation **(4 – 7)**:

$$u_{z,\mathrm{p}}(x) = C_3x^2 + C_4x + C_5 \tag{4 – 9}$$

By substituting equation **(4 – 9)** into the equation **(4 – 7)** values for the unknown coefficient C_3, C_4 and C_5 can be obtained:

$$C_3 = \frac{q}{2F}$$

$$C_4 = -\frac{qL}{2F}$$

$$C_5 = -\frac{q}{\alpha^2 F}$$

Using those coefficients and substituting the equation **(4 – 8)** into the equation **(4 – 7)**, coefficients C_1 and C_2 can be obtained using boundary conditions at the ends of the beam:

$$u_z(0) = 0 \rightarrow C_2 = \frac{q}{F\alpha^2} \tag{4 – 10}$$

$$u_z(L) = 0 \rightarrow C_1 = \frac{q}{F\alpha^2}\left(\frac{1-\cos(\alpha L)}{\sin(\alpha L)}\right) \tag{4 – 11}$$

Knowing the coefficients C_1, C_2, C_3, C_4 and C_5, the function for the beam deflection can be obtained from the equation **(4 – 8)**:

$$u_z(x) = \frac{q}{F\alpha^2}\left(\frac{1-\cos(\alpha L)}{\sin(\alpha L)}\sin(\alpha x) + \cos(\alpha x) - 1\right) + \frac{q}{2F}(x^2 - Lx) \tag{4 – 12}$$

Using goniometric formulae and evaluating the equation **(4 – 12)** at $x = \frac{L}{2}$, following expression for the maximum deflection of the beam can be obtained:

$$u_z\left(\frac{L}{2}\right) = \frac{q}{F\alpha^2}\left(2\frac{\sin^3(\frac{\alpha L}{2})}{\sin(\alpha L)} + \cos\left(\frac{\alpha L}{2}\right) - 1 - \frac{L^2\alpha^2}{8}\right) = 8.698 \text{ mm} \tag{4 – 13}$$

Stability

Critical load scaling factor can be defined as:

$$b = \frac{F_\mathrm{k}}{F} \tag{4 – 14}$$

where F_k is a critical load, which can be evaluated using formula **(4 – 7)** by neglecting the influence of the distributed loading and setting $F = F_\mathrm{k}$:

$$-EI\frac{\mathrm{d}^2u_z(x)}{\mathrm{d}x^2} = F_\mathrm{k}u_z(x) \tag{4 – 15}$$

which can be rewritten as follows:

$$\frac{\mathrm{d}^2u_z(x)}{\mathrm{d}x^2} + \beta^2u_z(x) = 0 \tag{4 – 16}$$

where $\beta^2 = \frac{F_\mathrm{k}}{EI}$. General solution of the equation **(4 – 16)** is:

$$u_z(x) = D\cos(\beta x) + E\sin(\beta x) \tag{4 – 17}$$

where D and E are coefficients, which can be determined using the beam's kinematic boundary conditions:

$$u_z(0) = 0 \rightarrow D = 0 \tag{4 – 18}$$

$$u_z(L) = 0 \rightarrow E \sin(\beta L) = 0 \tag{4 – 19}$$

The only non-trivial solution can be obtained if $\sin(\beta L) = 0$, which means that:

$$\beta L = n\pi \tag{4 – 20}$$

For the first mode of buckling is $n = 1$ and the critical load can be obtained from the equation **(4 – 20)** as follows:

$$F_{\mathrm{k}} = \frac{EI\pi^2}{L^2} \tag{4 – 21}$$

Substituting into the equation **(4 – 14)**, critical load scaling factor b can be expressed as:

$$b = \frac{EI\pi^2}{FL^2} = 9.868 \tag{4 – 22}$$

RFEM 5 and RSTAB 8 Settings

- Modeled in version RFEM 5.03.0050 and RSTAB 8.03.0050
- The element size is $l_{\mathrm{FE}} = 0.010$ m
- The number of increments is 1
- Shear stiffness of members is deactivated
- The Kirchhoff plate theory is used
- Isotropic linear elastic material model is used

Results

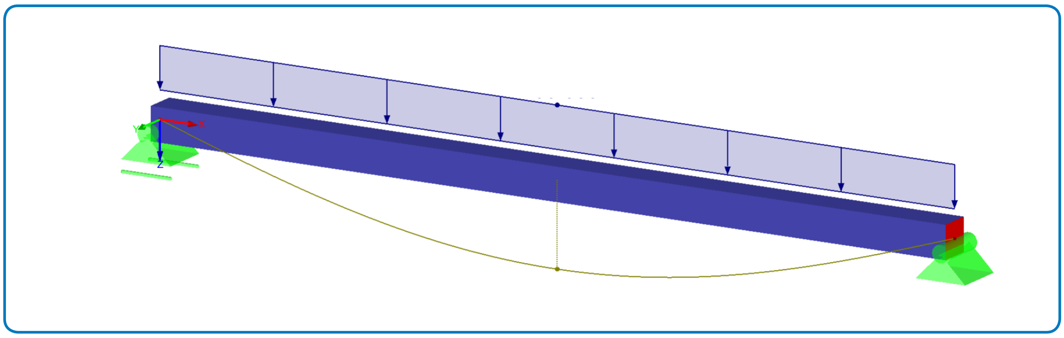

Figure 2: Results of the Geometrically linear analysis in RFEM 5

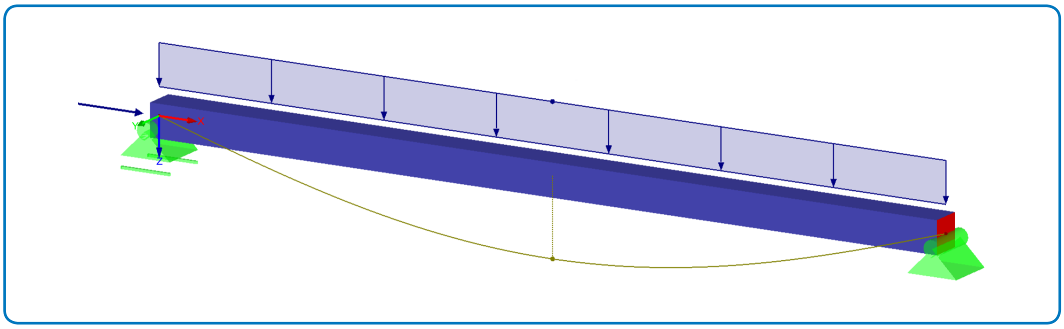

Figure 3: Results of the second-order analysis in RFEM 5

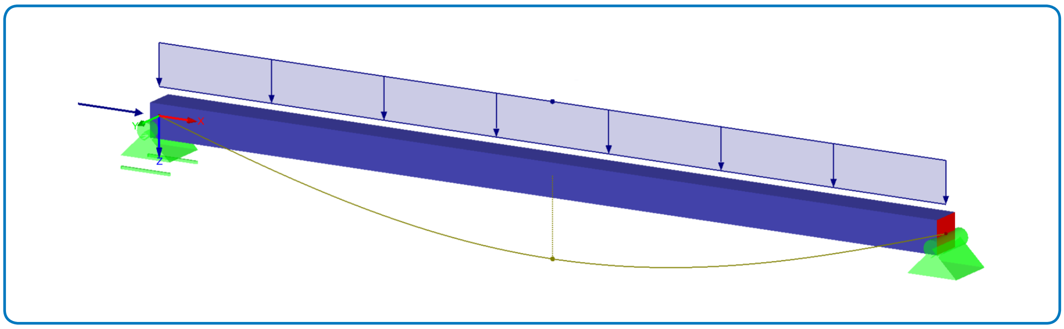

Figure 4: Results of the second-order analysis in RSTAB 8

Structure File	Program	Entity	Method of Analysis
0004.01	RFEM 5	Member	Geometrically Linear Analysis
0004.02	RFEM 5	Member	Second-Order Analysis
0004.03	RF-STABILITY	Member	Second-Order Analysis
0004.04	RFEM 5	Plate	Geometrically Linear Analysis
0004.05	RFEM 5	Plate	Second-Order Analysis
0004.06	RF-STABLITY	Plate	Second-Order Analysis
0004.07	RSTAB 8	Member	Geometrically Linear Analysis
0004.08	RSTAB 8	Member	Second-Order Analysis
0004.09	RSBUCK	Member	Second-Order Analysis

The following table shows the comparisons of the analytical method results with the outputs from RFEM 5 and RSTAB 8 considering the Geometrically linear analysis (no axial force effect) and the second-order analysis (with axial force effect) order theory:

Axial Force	Analytical Solution	RFEM 5 (Member)		RFEM 5 (Plate)		RSTAB 8 (Member)	
	$u_z(\frac{L}{2})$ [mm]	$u_z(\frac{L}{2})$ [mm]	Ratio [-]	$u_z(\frac{L}{2})$ [mm]	Ratio [-]	$u_z(\frac{L}{2})$ [mm]	Ratio [-]
No	7.813	7.813	1.000	7.813	1.000	7.813	1.000
Yes	8.698	8.697	1.000	8.699	1.000	8.697	1.000

The last table shows the comparisons of the critical load scaling factor b obtained by the linear stability analysis:

Axial Force	Analytical Solution	RF-STABILITY (Member)		RF-STABILITY (Plate)		RSBUCK (Member)	
	b [-]	b [-]	Ratio [-]	b [-]	Ratio [-]	b [-]	Ratio [-]
Yes	9.868	9.867	1.000	9.868	1.000	9.922	1.006