0103 – Natural Vibrations of a Planar Truss Structure

Description

A planar truss structure is simply supported according to the **Figure 1**. The aim of this verification example is to determine the natural frequencies of the structure. The problem is described by the following parameters.

Material	Steel	Modulus of Elasticity	E	210000.000	MPa
		Poisson's Ratio	ν	0.300	—
		Density	ρ	7850.000	kg/m³
Geometry		Field Width	w	0.500	m
		Field Height	h	0.700	m
		Cross-section Width	b	0.020	m

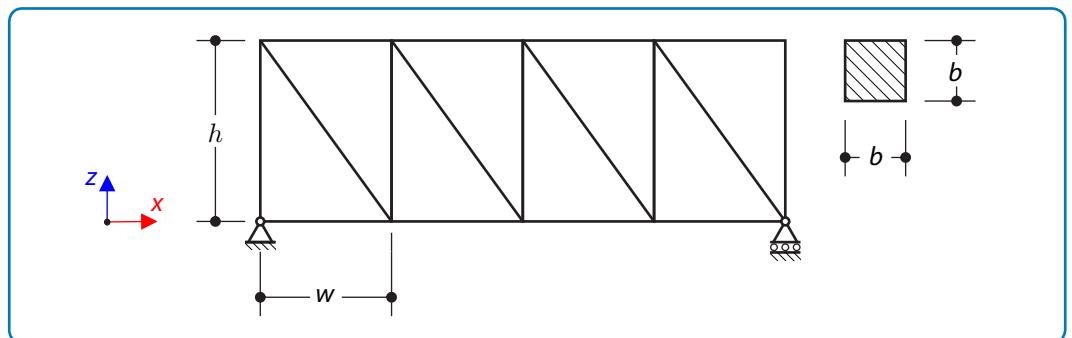Determine the natural frequencies of the truss structure.

Figure 1: Problem sketch

Analytical Solution

The eigenvalue problem of the planar truss construction can be conveniently solved by the finite element method (FEM). The truss structure is approximated by means of the truss elements. The stiffness matrix $\boldsymbol{k}^e$ and mass matrix $\boldsymbol{m}^e$ of the finite element is defined as, [1].

$$\boldsymbol{k}^e = \int_V \boldsymbol{B}^T \boldsymbol{E} \boldsymbol{B} \, dV \qquad (103 - 1)$$

$$\boldsymbol{m}^e = \int_V \boldsymbol{A}^T \boldsymbol{A} \, dV \qquad (103 - 2)$$

Where $\boldsymbol{A}$ is the matrix of the shape functions, $\boldsymbol{B}$ is the operator which defines the relationship between the strains and deformations[1], $\boldsymbol{E}$ is the matrix of material constants and dV is the volume

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[1] The operator $\boldsymbol{B}$ is defined as a derivative of the operator $\boldsymbol{A}$.

differential. The stiffness matrix $\boldsymbol{k}^\mathrm{e}$ and mass matrix $\boldsymbol{m}^\mathrm{e}$ for the linear truss element in plane can be calculated as follows.

$$\boldsymbol{k}^\mathrm{e} = \frac{EA}{L} \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad (103 - 3)$$

$$\boldsymbol{m}^\mathrm{e} = \frac{\rho AL}{6} \begin{bmatrix} 2 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 \\ 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 2 \end{bmatrix} \qquad (103 - 4)$$

Where A is the truss cross-section area and L is the truss length. The mass matrix defined in formula **(103 – 4)** is the consistent mass matrix. It can be modified into the diagonal mass matrix, which is used in the following calculation.

$$\boldsymbol{m}^\mathrm{e} = \frac{\rho AL}{6} \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix} \qquad (103 - 5)$$

The stiffness matrix of the element $\boldsymbol{k}^\mathrm{e}$ is valid only in the local coordinate system of the element. The mass matrix $\boldsymbol{k}^\mathrm{e}$ is invariant to the rotation of the element so that $\boldsymbol{m}^\mathrm{e} = \boldsymbol{m}$. The transformation into the global coordinate system can be done by the following transformation formula.

$$\boldsymbol{k} = \boldsymbol{T}^\mathsf{T} \boldsymbol{k}^\mathrm{e} \boldsymbol{T} \qquad (103 - 6)$$

Where $\boldsymbol{T}$ is the transformation matrix for the rotation of the element about the angle α.

$$\boldsymbol{T} = \begin{bmatrix} \cos\alpha & \sin\alpha & 0 & 0 \\ -\sin\alpha & \cos\alpha & 0 & 0 \\ 0 & 0 & \cos\alpha & -\sin\alpha \\ 0 & 0 & \sin\alpha & \cos\alpha \end{bmatrix} \qquad (103 - 7)$$

The local stiffness and mass matrices has to assembled into global mass matrix $\boldsymbol{M}$ and global stiffness matrix $\boldsymbol{K}$. The assembly process can be done by means of the code numbers which assign the global node number and global number of degrees of freedom to the local node numbers and local degrees of freedom. After that the system of second-order differential equation is obtained.

$$\boldsymbol{M}\frac{\mathrm{d}^2\boldsymbol{u}}{\mathrm{d}t^2} + \boldsymbol{K}\boldsymbol{u} = 0 \qquad (103 - 8)$$

Where $\boldsymbol{u}$ is the vector of the deformation. Furthermore the boundary conditions has to by applied. This should be done by means of matrices reduction. That is, in fact, elimination of the rows and columns which are relevant to the appropriate deformations. Expected the solution in the form $\boldsymbol{u} = \boldsymbol{U}e^{i\Omega t}$ the equation **(103 – 8)** can be written as follows.

$$(\boldsymbol{K} - \Omega^2 \boldsymbol{M})\, \boldsymbol{U} = \boldsymbol{0} \qquad\qquad (103 – 9)$$

Where $\boldsymbol{U}$ are eigenvectors and Ω^2 are natural angular frequencies. The formula **(103 – 9)** is obviously the eigenvalue problem, which can be solved. The natural frequencies can be obtained from the following definition.

$$f = \frac{\Omega}{2\pi} \qquad\qquad (103 – 10)$$

RFEM 5 Settings

- Modeled in RFEM 5.08.02
- The number of elements is one element for the truss
- Isotropic linear elastic material model is used
- Member entity is used
- Shear stiffness of the members is neglected

Results

Structure Files	Program
0103.01	RF-DYNAM Pro

First five natural frequencies are compared in the following table.

Natural Frequency	Analytical Solution	RF-DYNAM	Ratio
f_1 [Hz]	213.611	213.611	1.000
f_2 [Hz]	243.865	243.865	1.000
f_3 [Hz]	511.449	511.449	1.000
f_4 [Hz]	591.711	591.711	1.000
f_5 [Hz]	748.503	748.503	1.000

References

[1] STEJSKAL, V. and OKROUHLÍK, M. *Kmitání s Matlabem*. Vydavatelství ČVUT Praha.