# 0005 – Stability of a Beam with Various Support Stiffness

## Description

An axially loaded steel beam with a square cross-section is pinned at one end and spring supported at the other. Two cases with different spring stiffness $k_A$ and $k_B$ are considered. Neglecting its self-weight, determine the critical load scaling factors $b_A$ and $b_B$ by the linear stability analysis.

| | | | | | |
|---|---|---|---|---|---|
| Material | Steel | Modulus of Elasticity | $E$ | 200.00 | GPa |
| | | Poisson's Ratio | $\nu$ | 0.300 | – |
| Geometry | Beam | Height Width | $d$ | 0.010 | m |
| | | Length | $L$ | 1.000 | m |
| Spring | Case A | Stiffness | $k_A$ | 1.000 | kN/m |
| | Case B | Stiffness | $k_B$ | 2.000 | kN/m |
| Load | | Force | $F$ | 0.100 | kN |

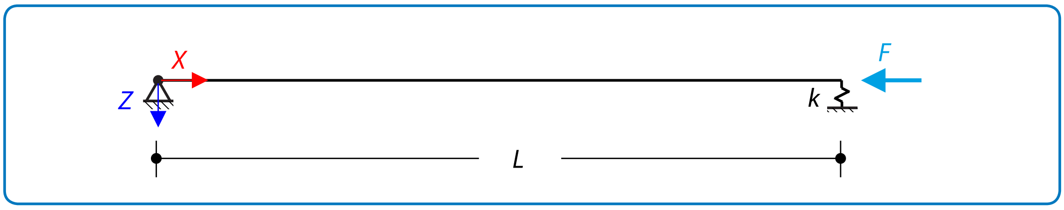


**Figure 1:** Problem sketch

## Analytical Solution

Critical load scaling factor $b$ can be defined as:

$$b = \frac{F_k}{F} \quad (5-1)$$

where $F_k$ is a critical load, which can be evaluated from the general solution of the differential equation for the deflection of the neutral axis:

$$u_z(x) = A + Bx + C\sin\left(\sqrt{\frac{F_k}{EI}}x\right) + D\cos\left(\sqrt{\frac{F_k}{EI}}x\right) \quad (5-2)$$

where $I = \frac{d^4}{12}$ is the second moment of the area. By setting $\alpha = \sqrt{\frac{F_k}{EI}}$ equation **(5 – 2)** can be rewritten as follows:

$$u_z(x) = A + Bx + C\sin(\alpha x) + D\cos(\alpha x) \quad (5-3)$$

Equation's **(5 – 3)** derivatives up to the third order can be given:

$$\frac{du_z}{dx} = B + C\alpha\cos(\alpha x) - D\alpha\sin(\alpha x) \quad (5-4)$$

$$\frac{d^2u_z}{dx^2} = -C\alpha^2\sin(\alpha x) - D\alpha^2\cos(\alpha x) \quad (5-5)$$

$$\frac{d^3u_z}{dx^3} = -C\alpha^3\cos(\alpha x) + D\alpha^3\sin(\alpha x) \quad (5-6)$$

Coefficients $A$, $B$, $C$, and $D$ can be evaluated from the following boundary conditions:

$$x = 0: \quad u_z = 0 \quad (5-7)$$

$$\frac{d^2u_z}{dx^2} = 0 \quad (5-8)$$

$$x = L: \quad u_z = 0 \quad (5-9)$$

$$\frac{d^3u_z}{dx^3} + \alpha\frac{du_z}{dx} - \beta u_z = 0 \quad (5-10)$$

where $\beta = \frac{k}{EI}$. Condition **(5 – 10)** can be obtained from the force equilibrium at the tip of the beam (Figure **Figure 2**) while expressing the shear force $V$ in the terms of the displacement $u_z$ and its derivatives:

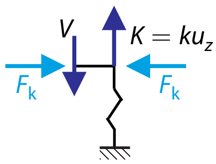


**Figure 2:** Force equilibrium at the tip of the beam

$$V = EI\frac{d^3u_z}{dx^3} + F_k\frac{du_z}{dx} \quad (5-11)$$

Substituting equations **(5 – 3)** - **(5 – 6)** into the equations **(5 – 7)** - **(5 – 10)** following four linear homogeneous equations can be obtained:

$$A + D = 0 \quad (5-12)$$

$$D = 0 \quad (5-13)$$

$$\alpha^2\sin(\alpha L)C = 0 \quad (5-14)$$

$$(\alpha^2 - \beta L)B - \sin(\alpha L)C = 0 \quad (5-15)$$

From equations **(5 – 13)** and **(5 – 12)** it is obvious that $D = 0$ and $A = 0$. To obtain non trivial solution, it will be considered that $B \neq 0$ and $C \neq 0$. By setting their determinant equal to zero following equation can be obtained:

$$\left(\frac{F_k}{kL} - 1\right) \sin \sqrt{\frac{F_k L^2}{EI}} = 0 \tag{5 – 16}$$

Following expressions for the critical force $F_k$ can be evaluated according to the equation **(5 – 16)**:

$$F_k = kL \tag{5 – 17}$$

$$F_k = \frac{\pi^2 EI}{L^2}, \frac{4\pi^2 EI}{L^2}, \frac{9\pi^2 EI}{L^2}, \ldots \tag{5 – 18}$$

To find the lowest critical force, it has to be determined if the following equation is valid:

$$k < \frac{\pi^2 EI}{L^3} = 1.645 \text{ kN/m} \tag{5 – 19}$$

For the Case A, where $k_A = 1.000$ kN/m $< 1.645$ kN/m, equation **(5 – 17)** should be considered and critical load scaling factor $b_A$ can be evaluated as follows:

$$b_A = \frac{kL}{F} = 10.000 \tag{5 – 20}$$

For the Case B, where $k_B = 2.000$ kN/m $> 1.645$ kN/m, equation **(5 – 18)** should be considered and critical load scaling factor $b_B$ can be evaluated as follows:

$$b_B = \frac{\pi^2 EI}{L^2 F} = 16.449 \tag{5 – 21}$$

## RFEM 5 and RSTAB 8 Settings

- Modeled in version RFEM 5.04.0058 and RSTAB 8.04.0058
- The element size is $l_{FE} = 0.100$ m
- The number of increments is 1
- Shear stiffness of members is deactivated
- Isotropic linear elastic material model is used

## Results

| Structure File | Program | Spring Stiffness |
|---|---|---|
| 0005.01 | RF-STABILITY | $k_a = 1$ kN/m |
| 0005.02 | RF-STABILITY | $k_b = 2$ kN/m |
| 0005.03 | RSBUCK | $k_a = 1$ kN/m |
| 0005.04 | RSBUCK | $k_b = 2$ kN/m |

As can be seen from the following comparisons, an excellent agreement of analytical solution with numerical output was achieved:

| Spring Stiffness | Analytical Solution | RF-STABILITY | | RSBUCK | |
|---|---|---|---|---|---|
| | b [-] | b [-] | Ratio [-] | b [-] | Ratio [-] |
| $k_a = 1$ kN/m | 10.000 | 10.000 | 1.000 | 10.000 | 1.000 |
| $k_b = 2$ kN/m | 16.449 | 16.449 | 1.000 | 16.450 | 1.000 |