0040 – Bending Console in Large Deformation

Description

A steel console is loaded by a single force on the right end and fully fixed on the left end ($u_x = u_y = u_z = \varphi_x = \varphi_y = \varphi_z = 0$) according to the **Figure 1**. The problem is described by the following set of parameters.

Material	Steel	Modulus of Elasticity	E	210.000	GPa
		Poisson's Ratio	ν	0.300	–
Geometry	Cantilever	Length	L	1.000	m
		Diameter	d	0.020	m
Load		Force	F	1.000	kN

The self-weight is neglected in this example. The loading force F is remaining vertical (it is not following the free end rotation). Determine the maximum deflection of the structure $u_{z,\max}$ and $u_{x,\max}$ by means of the large deformation analysis.

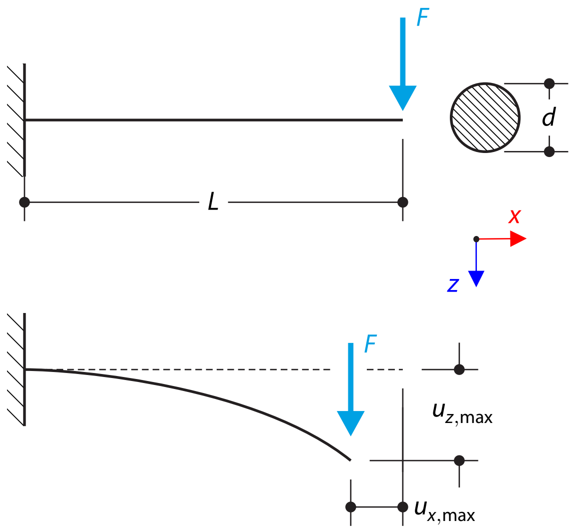

Figure 1: Problem Sketch and Solution

Analytical Solution

The problem can be analytically solved by the Ritz's method [1]. In the first step it is necessary to choose the right approximation function. The deflection in z-direction as a multiple of the linear solution is considered

$$u_z(x) = ax^2(3l_x - x) \qquad (40 – 1)$$

where a is an unknown constant. The first and the second derivative of the deflection can be calculated as

$$\frac{\mathrm{d}u_z(x)}{\mathrm{d}x} = \varphi_y(x) = 6al_x x - 3ax^2 \tag{40 – 2}$$

$$\frac{\mathrm{d}^2u_z(x)}{\mathrm{d}x^2} = 6al_x - 6ax \tag{40 – 3}$$

In this analytical approach the axial deflection is neglected (the length of the beam stays unchanged after deformation), which yields to the following approximation according to the **Figure 2**

$$\mathrm{d}u_z = \sin\varphi_y \ \mathrm{d}x \tag{40 – 4}$$

$$\mathrm{d}u_x = -(1 - \cos\varphi_y)\,\mathrm{d}x \tag{40 – 5}$$

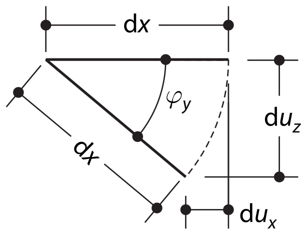

Figure 2: Deformed element

The energy of the internal forces is calculated as

$$E_{\text{int}} = \int_V \sigma\varepsilon \ \mathrm{d}V = \frac{EI_y}{2}\int_{l_x}\left(\frac{\mathrm{d}^2u_z(x)}{\mathrm{d}x^2}\right)^2 \mathrm{d}x = 6a^2l_x^3EI_y \tag{40 – 6}$$

where I_y is the quadratic moment of the cross-section to the y-axis[1]. The energy of the external forces is calculated as

$$E_{\text{ext}} = -F\int_{l_x} \mathrm{d}u_z = -F\int_{l_x} \sin\frac{\mathrm{d}u_z(x)}{\mathrm{d}x}\mathrm{d}x \tag{40 – 7}$$

In the following calculations the Taylor's polynomial series for the sine and cosine function at the point $x = 0$ is used.

$$\sin\varphi_y(x) = \varphi_y(x) - \frac{\varphi_y^3(x)}{3!} + O(\varphi_y^5(x)) \tag{40 – 8}$$

$$\cos\varphi_y(x) = 1 - \frac{\varphi_y^2(x)}{2!} + \frac{\varphi_y^4(x)}{4!} + O(\varphi_y^6(x)) \tag{40 – 9}$$

[1] $I_y = \frac{\pi d^4}{64} = 7854.000 \text{ mm}^4$

$$E_{\text{ext}} \approx -F \int_{l_x} \left(\frac{\mathrm{d}u_z(x)}{\mathrm{d}x} - \frac{\left(\frac{\mathrm{d}u_z(x)}{\mathrm{d}x}\right)^3}{6} \right) \mathrm{d}x = F\left(\frac{72}{35}a^3 l_x^7 - 2al_x^3\right) \tag{40 – 10}$$

The total potential energy is then

$$\Pi = E_{\text{int}} + E_{\text{ext}} = 6a^2 l_x^3 EI_y + F\left(\frac{72}{35}a^3 l_x^7 - 2al_x^3\right) \tag{40 – 11}$$

The unknown constant a is gotten by the principle of the minimum total potential energy of the system

$$\frac{\mathrm{d}\Pi}{\mathrm{d}a} = 0 \tag{40 – 12}$$

$$12al_x^3 EI_y + F\left(\frac{216}{35}a^2 l_x^7 - 2l_x^3\right) = 0 \tag{40 – 13}$$

$$a_1 = 9.805265 \cdot 10^{-8} \qquad a_2 = -3.305102650 \cdot 10^{-6}$$

The first solution a_1 is taken further. Due to the large deformations it is necessary to calculate both deflections by means of following integration. Taylor's polynomial series for the sine and cosine function is used again.

$$u_{z,\max} = \int_0^{l_x} \mathrm{d}u_z = \int_0^{l_x} \sin \varphi_y(x)\, \mathrm{d}x = 194.166 \text{ mm} \tag{40 – 14}$$

$$u_{x,\max} = \int_0^{l_x} \mathrm{d}u_x = -\int_0^{l_x} \left(1 - \cos \varphi_y(x)\right) \mathrm{d}x = -22.948 \text{ mm} \tag{40 – 15}$$

RFEM 5 and RSTAB 8 Settings

- Modeled in RFEM 5.04.0024 and RSTAB 8.04.0024
- The element size is $l_{\text{FE}} = 0.100$ m
- Large deformation analysis is considered
- The number of increments is 5
- The structure is modeled using members
- Shear stiffness of the members is neglected
- Isotropic linear elastic material model is used

Results

Structure File	Program
0040.01	RFEM 5
0040.02	RSTAB 8

	Analytical Solution	RSTAB 8	Ratio	RFEM 5	Ratio
$u_{z,\max}$ [mm]	194.166	194.241	1.000	194.240	1.000
$u_{x,\max}$ [mm]	−22.948	−22.935	0.999	−22.887	0.997

References

[1] BROŽOVSKÝ, J. and MATERNA, A. *Metoda konečných prvků ve stavební mechanice.* Ostrava, 2012.