0073 – Analysis of Plates Subjected to Different Load Types

Description

A simply supported rectangular plate with side lengths l_x and l_y is subjected to different load types. Assuming only small deformation theory and neglecting self-weight, determine the deflection at its centroid $u_z(\frac{l_x}{2}; \frac{l_y}{2})$ for each load type.

Material	Linear Elastic	Modulus of Elasticity	E	50.000	GPa
		Poisson's Ratio	ν	0.200	–
Geometry	Rectangle	Thickness	t	0.200	m
		Larger edge length	l_x	2.000	m
		Shorter edge length	l_y	1.000	m
Load	Uniform	Pressure	p_1	10.000	MPa
	Hydrostatic	Maximal Pressure	p_2	20.000	MPa
	Over a Part	Pressure	p_3	40.000	MPa
		x-position	x_3	1.500	m
		y-position	y_3	0.750	m
		x-dimension	d_{x3}	0.500	m
		y-dimension	d_{y3}	0.250	m
	Concentrated	Force	F_4	50.000	MN
		x-position	x_4	1.500	m
		y-position	y_4	0.750	m

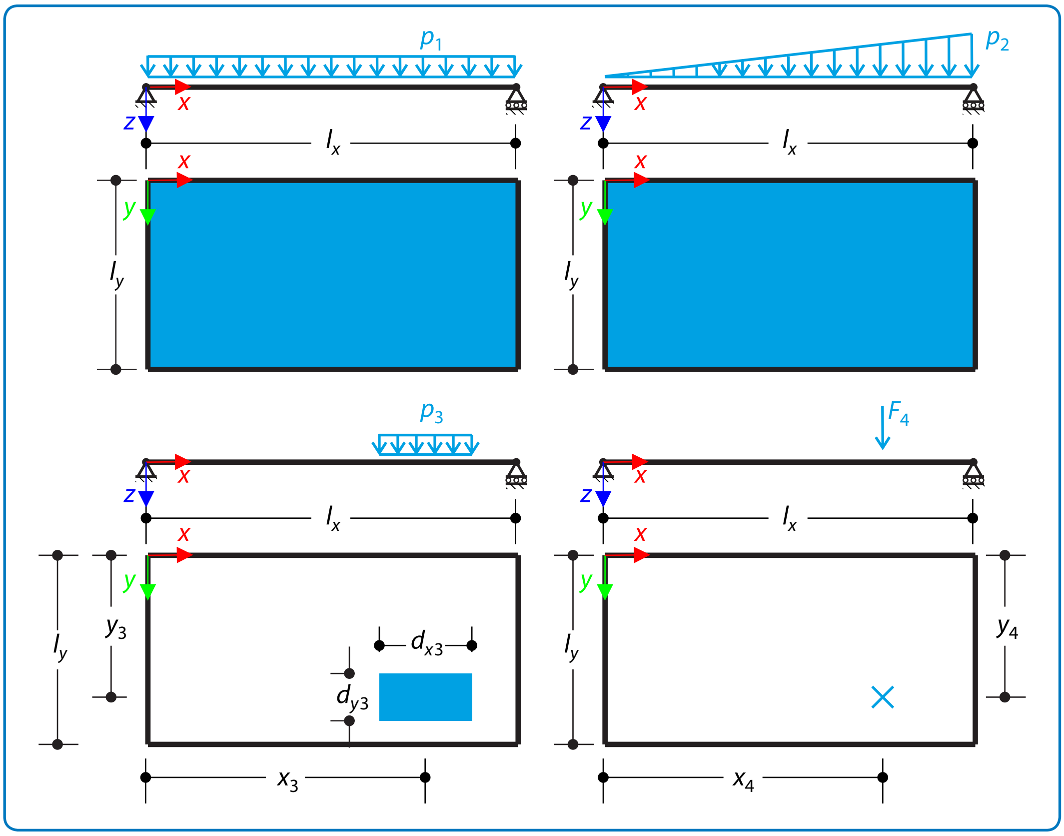

Figure 1: Problem Sketch

Analytical Solution

Governing differential equation of the plate subjected to the transversal load $p_z(x, y)$ has the following form [1]:

$$\frac{\partial^4 u_z}{\partial x^4} + 2\frac{\partial^4 u_z}{\partial x^2 \partial y^2} + \frac{\partial^4 u_z}{\partial y^4} = \frac{p_z(x, y)}{D} \qquad (73 – 1)$$

where D is the flexural rigidity of the plate:

$$D = \frac{Et^3}{12(1 - \nu^2)} \qquad (73 – 2)$$

By Navier's method the deflected plate surface $u_z(x, y)$ and the transversal load $p_z(x, y)$ can be expressed by a double Fourier series with coefficients U_{mn} and P_{mn}:

$$u_z(x, y) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} U_{mn} \sin\frac{m\pi x}{l_x} \sin\frac{n\pi y}{l_y} \qquad (73 – 3)$$

$$p_z(x, y) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} P_{mn} \sin\frac{m\pi x}{l_x} \sin\frac{n\pi y}{l_y} \qquad (73 – 4)$$

U_{mn} can be obtained by substituting **(73 – 3)** and **(73 – 4)** into **(73 – 1)**:

$$U_{mn} = \frac{P_{mn}}{D\pi^4 \left(\frac{m^2}{l_x^2} + \frac{n^2}{l_y^2}\right)^2} \qquad (73 – 5)$$

Substituting **(73 – 5)** into **(73 – 3)**, an analytical solution for the deflection of the plate can be obtained:

$$u_z(x,y) = \frac{1}{D\pi^4} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{P_{mn}}{\left(\frac{m^2}{l_x^2} + \frac{n^2}{l_y^2}\right)^2} \sin \frac{m\pi x}{l_x} \sin \frac{n\pi y}{l_y} \qquad (73-6)$$

Uniformly Distributed Pressure

When the plate is subjected to uniformly distributed pressure, the coefficient P_{mn} takes the following form:

$$P_{mn} = \frac{16 p_1}{\pi^2 mn} \qquad (73-7)$$

for positive odd integers m and n. Substituting **(73 – 7)** into **(73 – 6)**, the deflection surface of the simply supported rectangular plate under uniformly distributed pressure can be derived:

$$u_{z1}(x,y) = \frac{16 p_1}{D\pi^6} \sum_{m=1,3,5,\ldots} \sum_{n=1,3,5,\ldots} \frac{1}{mn\left(\frac{m^2}{l_x^2} + \frac{n^2}{l_y^2}\right)^2} \sin \frac{m\pi x}{l_x} \sin \frac{n\pi y}{l_y} \qquad (73-8)$$

Finally, evaluation at the plate centroid:

$$u_{z1}\left(\frac{l_x}{2}, \frac{l_y}{2}\right) \approx 2.916 \text{ mm} \qquad (73-9)$$

Hydrostatic Pressure

When the plate is subjected to hydrostatic pressure, P_{mn} takes the following form:

$$P_{mn} = \frac{4}{l_x l_y} \int_0^{l_x} \int_0^{l_y} p_z(x,y) \frac{m\pi x}{l_x} \frac{n\pi y}{l_y} \mathrm{d}x\mathrm{d}y \qquad (73-10)$$

Knowing that $p_z = \frac{p_2 x}{l_x}$, the equation **(73 – 10)** can be rewritten as follows:

$$P_{mn} = \frac{4 p_2}{l_x^2 l_y} \int_0^{l_x} x \sin \frac{m\pi x}{l_x} \mathrm{d}x \int_0^{l_y} \sin \frac{n\pi y}{l_y} \mathrm{d}y \qquad (73-11)$$

Evaluating the integral in **(73 – 11)** the coefficient P_{mn} can be significantly simplified:

$$P_{mn} = -\frac{8 p_2 \cos m\pi}{mn\pi^2} \qquad (73-12)$$

By substituting **(73 – 12)** into **(73 – 6)**, the deflection surface of the simply supported rectangular plate subjected to the hydrostatic pressure can be derived:

$$u_{z2}(x,y) = \frac{8p_2}{D\pi^6} \sum_{m=1,2,3,\dots} \sum_{n=1,2,3,\dots} \frac{(-1)^{m+1}}{mn\left(\frac{m^2}{l_x^2}+\frac{n^2}{l_y^2}\right)^2} \sin\frac{m\pi x}{l_x} \sin\frac{n\pi y}{l_y} \tag{73 – 13}$$

Deflection of the plate at its centroid can be easily evaluated as:

$$u_{z2}\left(\frac{l_x}{2},\frac{l_y}{2}\right) \approx 2.916 \text{ mm} \tag{73 – 14}$$

Uniformly Distributed Pressure Over Part of the Plate

When only a part of the plate is subjected to uniformly distributed pressure, the coefficient P_{mn} takes the following form:

$$P_{mn} = \frac{4p_3}{l_x l_y} \int_{x_3-d_{x3}/2}^{x_3+d_{x3}/2} \int_{y_3-d_{y3}/2}^{y_3+d_{y3}/2} \sin\frac{m\pi x}{l_x} \sin\frac{m\pi y}{l_y} \mathrm{d}x\mathrm{d}y \tag{73 – 15}$$

which evaluates as:

$$P_{mn} = \frac{16p_3}{\pi^2 mn} H_{mn} \tag{73 – 16}$$

where

$$H_{mn} = \sin\frac{m\pi x_3}{l_x} \sin\frac{n\pi y_3}{l_y} \sin\frac{m\pi d_{x3}}{2l_x} \sin\frac{n\pi d_{y3}}{2l_y} \tag{73 – 17}$$

for all integers m and n. By substituting **(73 – 16)** into **(73 – 6)**, the deflection surface of the simply supported rectangular plate having its part subjected to the uniformly distributed pressure can be derived:

$$u_{z3}(x,y) = \frac{16p_3}{D\pi^6} \sum_{m=1,2,3,\dots} \sum_{n=1,2,3,\dots} \frac{H_{mn}}{mn\left(\frac{m^2}{l_x^2}+\frac{n^2}{l_y^2}\right)^2} \sin\frac{m\pi x}{l_x} \sin\frac{n\pi y}{l_y} \tag{73 – 18}$$

Deflection of the plate at its centroid can be easily evaluated as:

$$u_{z3}\left(\frac{l_x}{2},\frac{l_y}{2}\right) \approx 0.776 \text{ mm} \tag{73 – 19}$$

Concentrated Force

When the plate is subjected to the concentrated force, P_{mn} can be derived analogously to the previous case considering that $p_4 = \frac{F_4}{d_{x4}d_{y4}}$ where $d_{x4} \to 0$ and $d_{y4} \to 0$:

$$P_{mn} = \frac{16F_4}{\pi^2 mnd_{x4}d_{y4}} \sin\frac{m\pi x_4}{l_x} \sin\frac{n\pi y_4}{l_y} \sin\frac{m\pi d_{x4}}{2l_x} \sin\frac{n\pi d_{y4}}{2l_y} \quad (73-20)$$

However, to be able to apply the limit approach, the equation **(73 – 20)** must be modified. By multiplying and dividing the equation **(73 – 20)** by $l_x \times l_y$ the following expression can be obtained:

$$P_{mn} = \lim_{l_x \to 0, l_y \to 0} \left(\frac{4F_4}{l_x l_y} sin \frac{m\pi x_4}{l_x} sin \frac{n\pi y_4}{l_y} \frac{sin\frac{m\pi d_{x4}}{2l_x} sin\frac{n\pi d_{y4}}{2l_y}}{\frac{m\pi d_{x4}}{2l_x}\frac{n\pi d_{y4}}{2l_y}} \right) \quad (73-21)$$

which after modification becomes

$$P_{mn} = \frac{4F_4}{l_x l_y} sin \frac{m\pi x_4}{l_x} sin \frac{n\pi y_4}{l_y} \quad (73-22)$$

By substituting **(73 – 22)** into **(73 – 6)**, the deflection surface of the simply supported rectangular plate subjected to the concentrated force can be derived:

$$u_{z4}(x,y) = \frac{4F_4}{D\pi^4 l_x l_y} \sum_{m=1,2,3,\ldots} \sum_{n=1,2,3,\ldots} \frac{sin\frac{m\pi x_4}{l_x} sin\frac{n\pi y_4}{l_y}}{\left(\frac{m^2}{l_x^2} + \frac{n^2}{l_y^2}\right)^2} \sin\frac{m\pi x}{l_x} \sin\frac{n\pi y}{l_y} \quad (73-23)$$

Deflection of the plate at its centroid can be easily evaluated as:

$$u_{z4}\left(\frac{l_x}{2}, \frac{l_y}{2}\right) \approx 7.848 \text{ mm} \quad (73-24)$$

RFEM 5 Settings

- Modeled in version RFEM 5.07.01
- Element size is $l_{FE} = 0.010$ m
- Geometrically linear analysis is considered
- Number of increments is 1
- Kirchhoff plate theory is used

Results

Structure File	Load Type
0073.01	Uniformly Distributed Pressure
0073.02	Hydrostatic Pressure
0073.03	Uniformly Distributed Pressure Over Part of Plate
0073.04	Concentrated Force

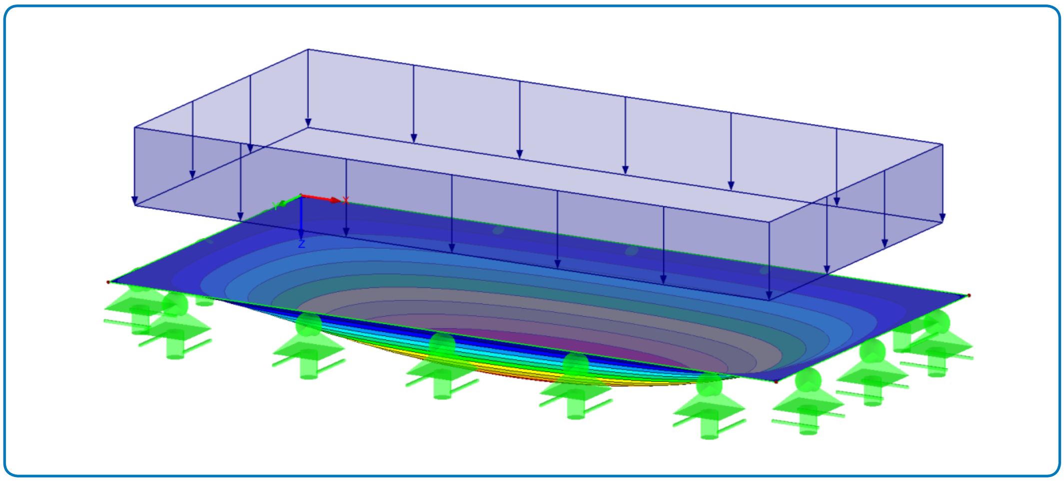

Figure 2: Uniformly Distributed Pressure

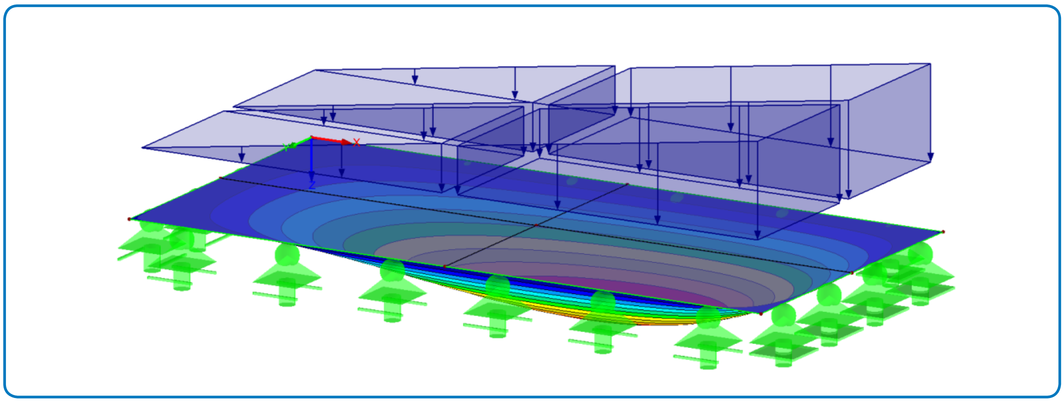

Figure 3: Hydrostatic Pressure

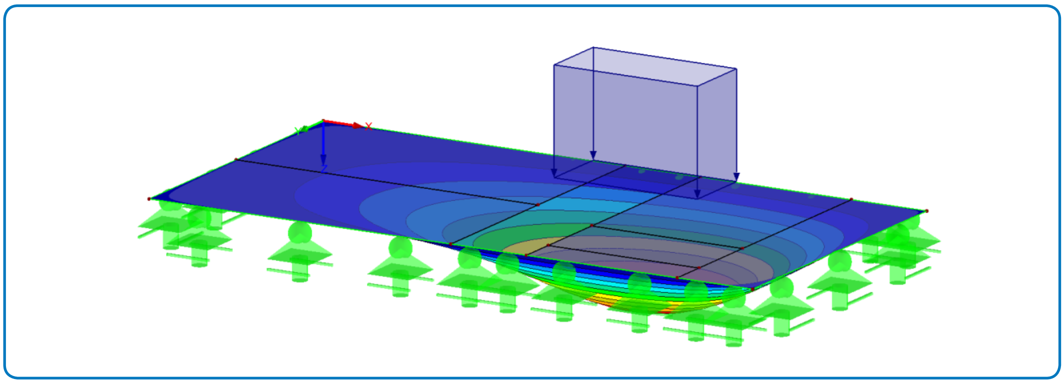

Figure 4: Uniformly Distributed Pressure Over Part of the Plate

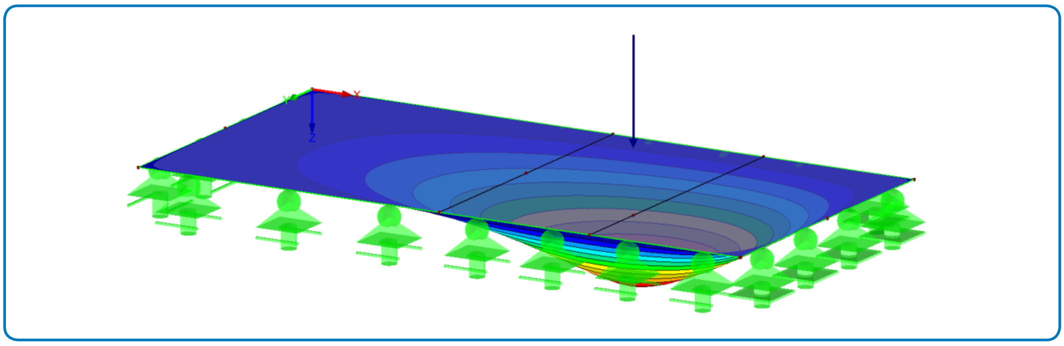

Figure 5: Concentrated Pressure

Load Type	Analytical Solution	RFEM 5	
	$u_z\left(\frac{l_x}{2}, \frac{l_y}{2}\right)$ [mm]	$u_z\left(\frac{l_x}{2}, \frac{l_y}{2}\right)$ [mm]	Ratio [-]
Uniformly Distributed Load	2.916	2.917	1.000
Hydrostatic Pressure	2.916	2.917	1.000
Uniformly Distributed Partial Pressure	0.776	0.776	1.000
Concentrated Force	7.848	7.848	1.000

References

[1] SZILARD, R. *Theories and Application of Plate Analysis: Classical Numerical and Engineering Method*. Hoboken, New Jersey.