# Consideration of the P-Δ Effect in Dynamic Analysis

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## Requirements of ASCE 7

For the ultimate limit state design, **ASCE 7-16 Section 12.8.7** [1] requires second-order theory calculation with include P-Δ effects. However, this effect can be disregarded when the stability coefficient $\theta$ is less than or equal to $0.1$. $\theta$ is defined by the following:

$$\theta = \frac{P_x \cdot \Delta \cdot l_e}{V_x \cdot h_{sx} \cdot C_d} \tag{1}$$

Where,

| | |
|---|---|
| $\theta$ | stability coefficient |
| $P_x$ | total vertical design load at and above level x [kip (kN)]; when computing $P_x$, no individual load factor need to exceed 1.0 |
| $\Delta$ | design story drift as defined in Section 12.8.6 |
| $l_e$ | importance factor determined in accordance with Section 11.5.1 and Table 1.5-2 |
| $V_x$ | seismic shear force acting between levels x and x-1 [kip (kN)] |
| $h_{sx}$ | story height below level x [in (mm)], and |
| $C_d$ | deflection amplification factor in Table 12.2-1 |

$\theta$ should not exceed $\theta_{max}$ determined by the following:

$$\theta_{max} = \frac{0.5}{\beta \cdot C_d} \leq 0.25 \tag{2}$$

Where $\beta$ is the ratio of shear demand to shear capacity for the story between levels x and x-1. This ratio can conservatively be taken as 1.0.
When $0.1 < \theta \leq$ θmax, second-order effects may be approximated by the factor $1/(1-\theta)$. If $\theta > \theta_{max}$, the structure may be unstable and requires redesign.

## Geometric Stiffness Matrix

For dynamic analyses, nonlinear iterative calculations for second-order theory are not applicable. Rather, the analysis can be linearized with the geometric stiffness matrix based on axial loads only to account for second-order effects. It is assumed the vertical loads do not change due to horizontal effects and the deformations are small compared to the structure's overall dimensions [2]. The loads considered should correspond to the seismic load combinations specified in the **ASCE 7-16 Section 2.3.6** [1]:

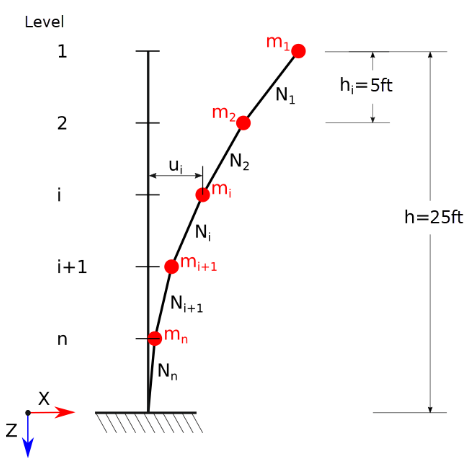
$$[6] \quad 1.2D + E_v + E_h + L + 0.2S \tag{3}$$

$$[7] \quad 0.9D - E_v + E_h \tag{4}$$

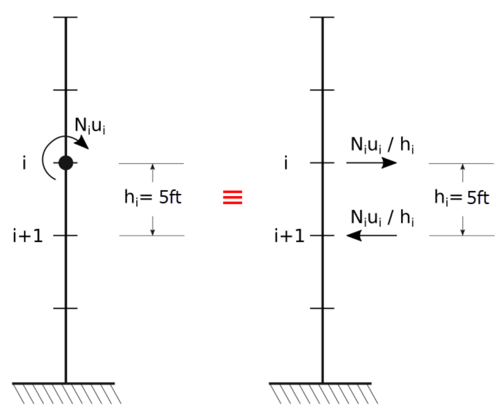
Where,

| | |
|---|---|
| $D$ | dead load |
| $E_v$ | vertical seismic forces, applied in the downward direction, as defined in Section 12.4.2.2 |
| $E_h$ | horizontal seismic forces as defined in Section 12.4.2.1 |
| $L$ | live load |
| $S$ | snow load |

For members such as a prestressed cable, axial tensile forces increase the stiffness. In turn, compression forces reduce the stiffness and can lead to a singularity in the stiffness matrix. The geometric stiffness matrix, $\mathbf{K_g}$, is independent from the structure's mechanical properties and depends only on the member length ($l$) and axial force ($N$).
To illustrate this basic concept, an example of a cantilever is displayed in Figure 1. The single mass points on the cantilever represent the individual building stories. The building is subjected to a dynamic analysis considering the second-order theory. The axial forces, $N_i$, on the individual storeys $i = 1, 2, \ldots, n$ result from the vertical forces in the seismic design situation (see Equations 3 and 4). The story height is defined by $h_i$.




(a) Displaced position of normal forces $N_i$




(b) Resulting overturning moments / lateral loads

**Figure 1:** Simplification of a building to a cantilever structure. The individual mass points represent stories. Displacement due to compression normal forces shown in (a) is converted to overturning moments and lateral loads (b)[2].

$\mathbf{K_g}$ can be derived from the static equilibrium conditions:

$$\begin{bmatrix} F_i \\ F_{i+1} \end{bmatrix} = \underbrace{\frac{N_i}{h_i} \begin{bmatrix} 1.0 & -1.0 \\ -1.0 & 1.0 \end{bmatrix}}_{\mathbf{K_g}} \begin{bmatrix} u_i \\ u_{i+1} \end{bmatrix} \tag{5}$$

For simplification, only the horizontal displacement degrees of freedom are displayed. The derivation is based on the approach of overturning moments with the assumption of linear displacement. This is a further simplification for the bending element but an accurate theory for the truss element. More precise determination of the geometric stiffness matrix for bending beams can be obtained by using the cubic displacement approach or the analytical solution of the bending line differential equation. More information and derivations are provided by Werkle [3].

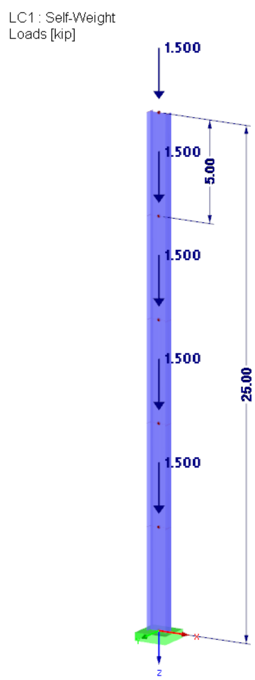
$\mathbf{K_g}$ is added to the system stiffness matrix, $\mathbf{K}$, to obtain the modified stiffness matrix, $\mathbf{K_{mod}}$, where:

$$\mathbf{K_{mod}} = \mathbf{K} + \mathbf{K_g} \tag{6}$$

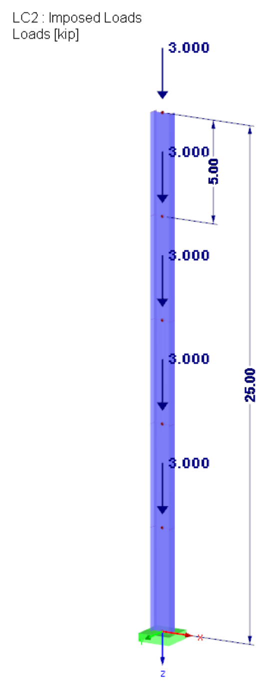
In the case of compression normal forces, this consequently leads to the stiffness reduction.

## Example: Natural Frequencies and Multi-Modal Response Spectrum Analysis Considering Second-Order Theory

The following example shows how the geometric stiffness matrix can be considered in *RFEM* and the *RF-DYNAM Pro* add-on modules. The cantilever in Figure 1 consists of five concentrated mass points. A value of $6276.2\ lb$ act in the global X-direction at each mass point. The self-weight and the additional loads are applied as nodal loads in two separate load cases illustrated in Figure 2.




(a) Self-Weight in LC1




(b) Imposed Loads in LC2

**Figure 2:** Self-weight and imposed loads are applied as nodal loads in two separate load cases.

The member is a W 12x26 cross-section and *A992* material with $I_y = 204\ in^4$ and $E = 29000\ ksi$.
In order to consider the geometric stiffness matrix in the dynamic analysis, a load combination is initially defined for the seismic design situation in the main program *RFEM* (Eqn. 3). In this example, only Equation 3 is examined as this has the most unfavorable effects on the structure. The definition of the *Load Combination* and the resulting axial forces are shown in Figure 3.

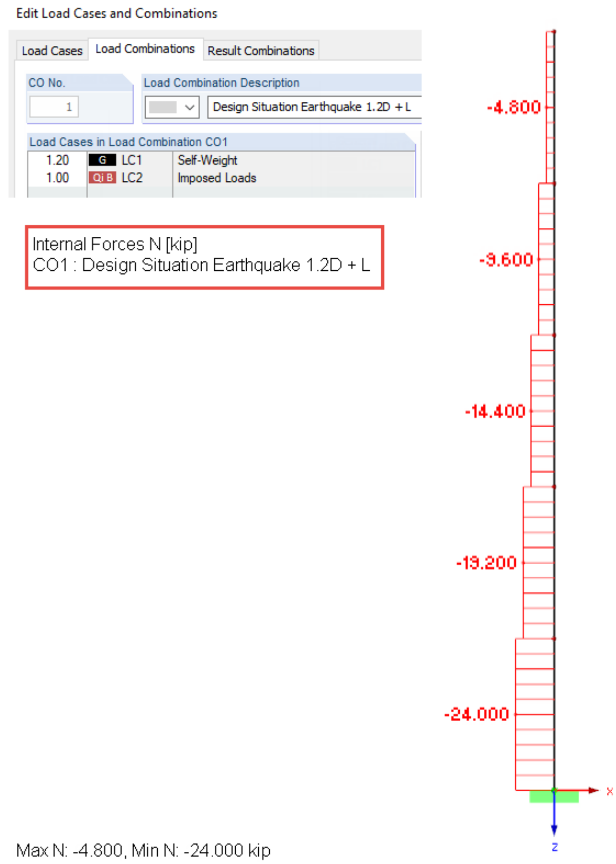



**Figure 3:** Definition of the load combination for the seismic design situation (Equation 3) and the resulting axial forces. These axial forces are used to determine the geometric stiffness matrix.

The *RF-DYNAM Pro – Natural Vibrations* add-on module determines the natural frequencies, mode shapes and effective modal masses of a structure while taking into account various stiffness modifications (see *RF-DYNAM Pro* Manual, Section 2.4.7 [4] and in the Knowledge Base [5]).

In this example, two mass cases are defined by importing masses from the load cases. These are combined into mass combinations with the combination factors in accordance with **ASCE 7-16 Section 12.7.2** [1].

Two natural vibration cases are defined in the *RF-DYNAM Pro – Natural Vibrations* module. *NVC1* is defined which does not include any stiffness modifications. For *NVC2*, *CO1* is imported to consider the geometric stiffness matrix and, in turn, the second-order theory. The definition of the *Natural Vibration Cases* is illustrated in Figure 4.

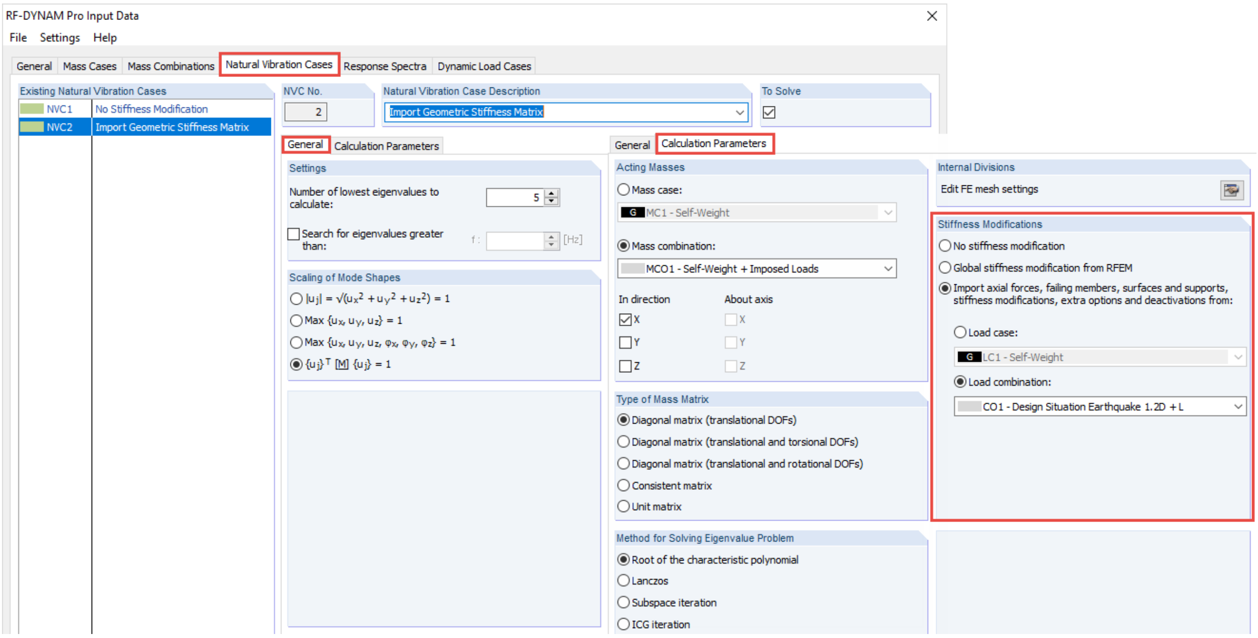



**Figure 4:** Parameters for Eigenvalue Analysis in *RF-DYNAM Pro - Natural Vibrations*. In the natural vibration case *NVC2*, the *CO1* is imported to take the P-$\Delta$ effect into account. For comparison reasons, *NVC1* is defined without any stiffness modifications.

Table 1 includes the determined natural frequencies $f$ $[Hz]$ and the natural periods $T$ $[sec]$ - with and without the geometric stiffness matrix consideration.

| Mode Shape | Without stiffness modifications (*NVC1*) Frequency $f$ $[Hz]$ | Period $T$ $[sec]$ | Spectrum $S_a$ $[ft/s^2]$ | With geometric stiffness matrix (*NVC2*) Frequency $f$ $[Hz]$ | Period $T$ $[sec]$ | Spectrum $S_a$ $[ft/s^2]$ |
|---|---|---|---|---|---|---|
| 1 | 1.290 | 0.775 | 2.563 | 1.250 | 0.800 | 2.461 |
| 2 | 8.239 | 0.121 | 8.590 | 8.203 | 0.122 | 8.613 |
| 3 | 23.340 | 0.043 | 5.155 | 23.303 | 0.043 | 5.158 |
| 4 | 45.089 | 0.022 | 4.251 | 45.051 | 0.022 | 4.252 |
| 5 | 67.123 | 0.015 | 3.933 | 67.086 | 0.015 | 3.933 |

**Table 1:** Natural Frequencies $f$ $[Hz]$, natural periods $T$ $[sec]$, and resulting acceleration values $S_a$ $[ft/s^2]$ read from the response spectrum - with and without $\mathbf{K_g}$ resulting from the *CO1* axial forces.

The multi-modal response spectrum analysis uses natural frequencies to determine the acceleration values from the defined response spectrum. Based on these acceleration values, the equivalent loads and the response spectrum internal forces are determined. The graphic illustration of a user-defined response spectrum is shown in Figure 5. The acceleration values, $S_a$, determined from the response spectrum for each eigenvalue are listed in Table 1. In order to ensure the correct allocation of the modified frequencies, the corresponding natural vibration case (NVC) must be assigned to the dynamic load case (DLC). This setting is shown in Figure 6.

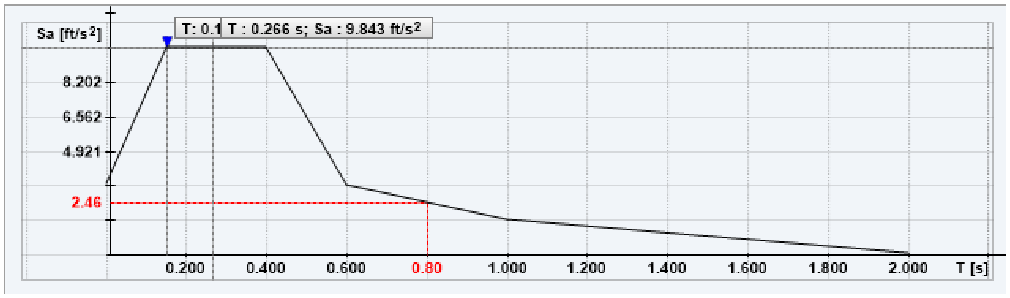



**Figure 5:** User-Defined Response Spectrum.

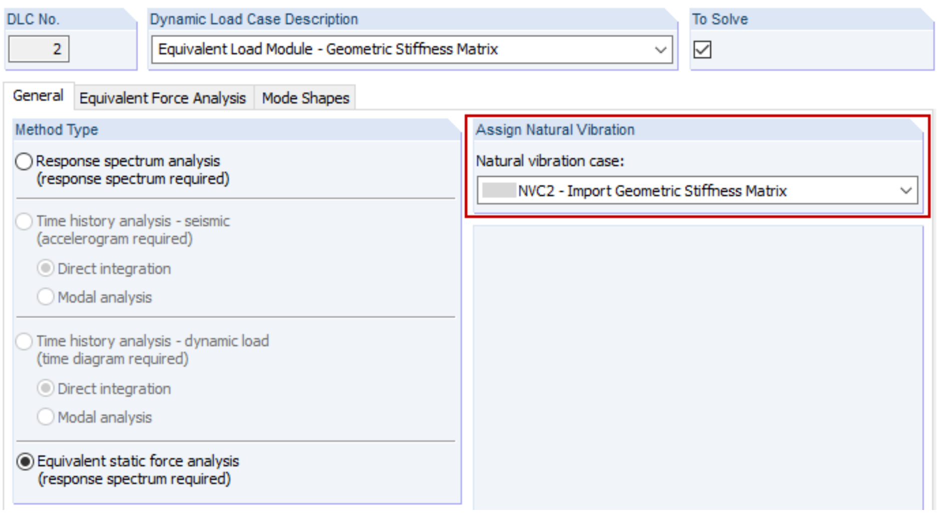



**Figure 6:** Assignment of the *Natural Vibration Case* to the *Dynamic Load Case* to determine the equivalent loads.

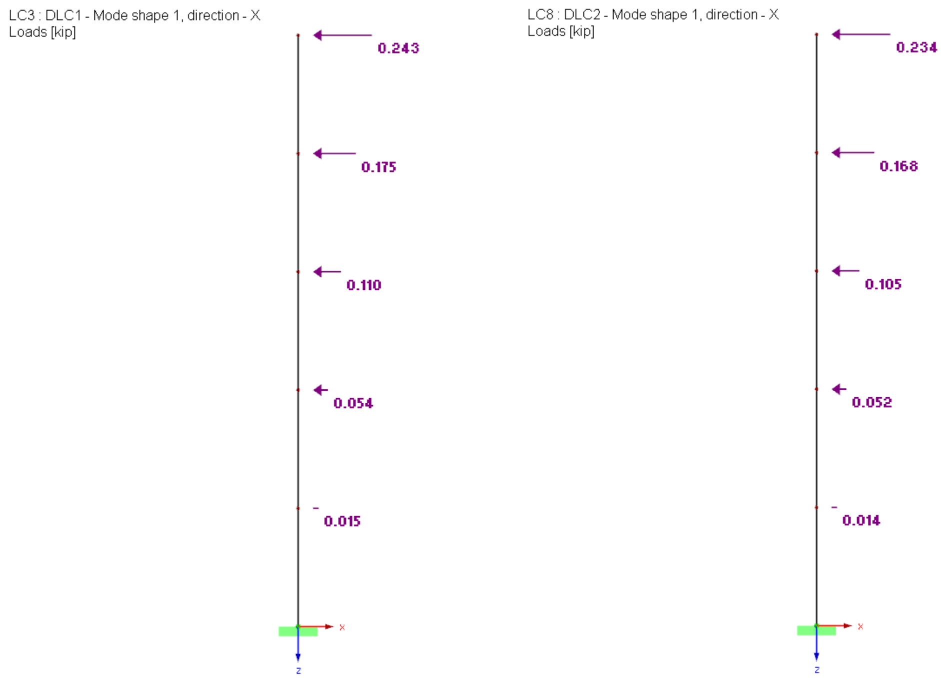
In the case of compression axial forces, considering the geometric stiffness matrix leads to a reduced natural frequency and can cause lower corresponding acceleration values, $S_a$. This can be seen in the current example. Modification of the natural frequencies is not sufficient to consider P-$\Delta$ effects. In fact, this may actually lead to smaller results, which may be incorrect. It is important to use the modified geometric stiffness matrix to determine the internal forces and deformations.

In *RF-DYNAM Pro – Forced Vibrations*, the modified stiffness is automatically used to determine the response spectrum results because the calculation is performed within *RF-DYNAM Pro*.

In *RF-DYNAM Pro – Equivalent Loads*, the equivalent loads are determined and exported as load cases into the main program *RFEM*. Therefore, the calculation is performed partially in *RF-DYNAM Pro* and partially in *RFEM*. Theoretical background for the equivalent load calculation is explained in the Manual of *RF-DYNAM Pro* [4]. The Verification Example [6] shows the calculation for a specific example. The determined equivalent loads - with and without the geometric stiffness matrix - are displayed in Figure 7.

Exporting the equivalent loads has many advantages, but it is important to transfer the stiffness modifications correctly into the corresponding load cases. The calculation parameters of the exported load cases must be adjusted as shown in Figure 8.

The individual load cases are superimposed using the *SRSS* or *CQC* method. This is automatically set in *RF-DYNAM Pro* and exported as result combinations in *RFEM*. The results with and without the geometric stiffness matrix are displayed in Figure 9.




(a) No Stiffness Modifications (b) With geometric stiffness matrix

**Figure 7:** Equivalent loads for mode shape 1 without stiffness modifications from *DLC1* (a) and with geometric stiffness matrix from *DLC2* (b).

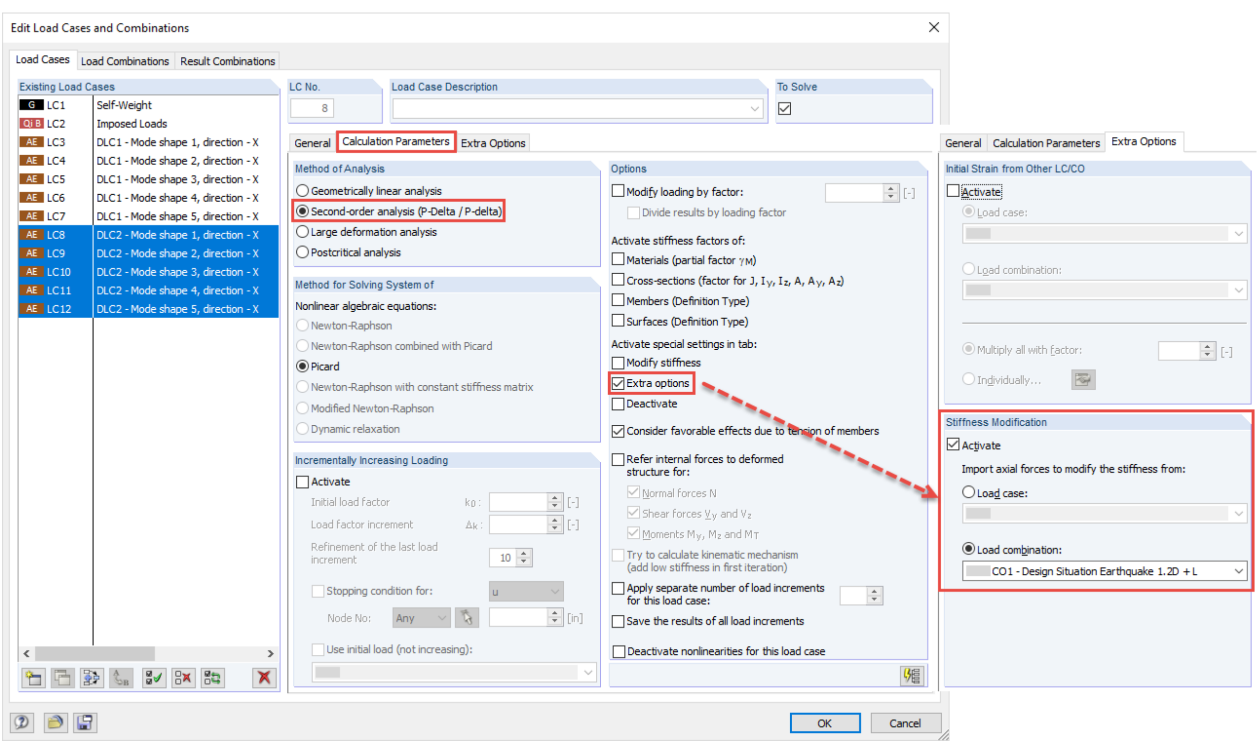
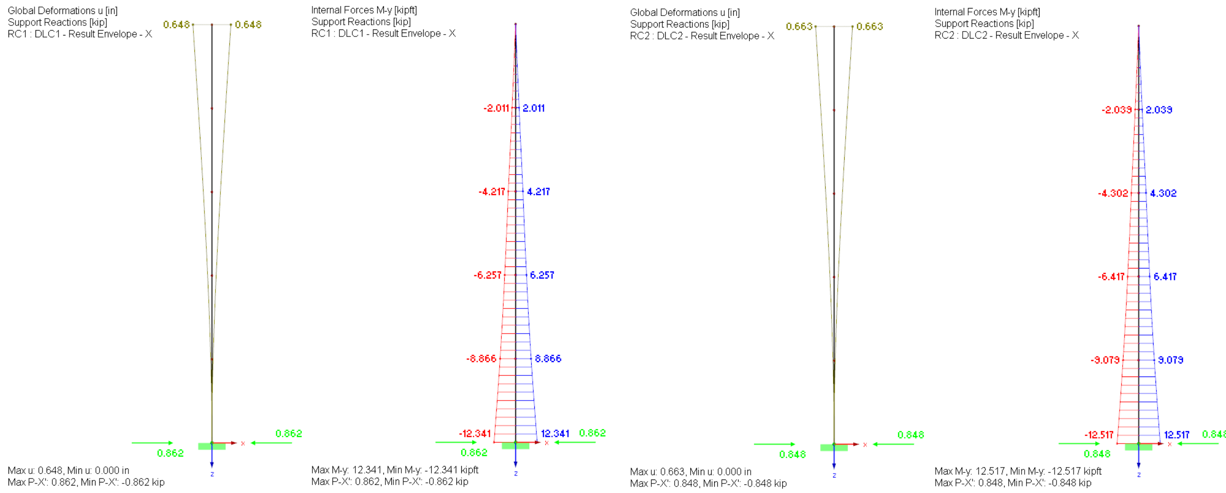



**Figure 8:** Calculation parameters of load cases with the exported equivalent loads. Here the geometric stiffness matrix must be considered by importing the axial forces from *CO1*.




(a) No Stiffness Modifications

(b) With geometric stiffness matrix

**Figure 9:** Deformations ($u_X$), moment ($M_Y$), and support reactions ($P_X$) resulting from the multi-modal response spectrum analysis with no stiffness modifications from *DLC1* (a) and with the geometric stiffness matrix from *DLC2* (b).

Considering the geometric stiffness matrix leads to larger deformations and internal forces. However, the equivalent loads and resulting support loads are slightly smaller when considering the geometric stiffness matrix.

## References

[1] ASCE 7-16. *Mimimum Design Loads and Associated Criteria for Buildings and Other Structures*, 2017.

[2] Edward L. Wilson. *Three-dimensional static and dynamic analysis of structures*. CSi - Computers and Structures Inc., 2002.

[3] Horst Werkle. *Finite Elemente in der Baustatik - Statik und Dynamik in der Stab- und Flächentragwerke*. Vieweg, 3rd edition, 2008 (in German).

[4] Dlubal Software GmbH. *Add-on Module RF-DYNAM Pro - Program Description*, 2018.

[5] Gerlind Schubert. *Import of Stiffness Modifications in RF-DYNAM Pro - Natural Vibrations*. 2015. URL `https://www.dlubal.com/en-US/support-and-learning/support/knowledge-base/001023`.

[6] Dlubal Software GmbH. *Verification Example 105: Equivalent Loads*. 2015. URL `https://www.dlubal.com/en-US/downloads-and-information/examples-and-tutorials/verification-examples/0105`.