Program: RFEM 5, RF-LAMINATE, RF-GLASS

Category: Geometrically Linear Analysis, Isotropic Linear Elasticity, Glass, Laminate, Plate

Verification Example: 0028 – Double-Pane Glass Without Coupling of Layers

0028 – Double-Pane Glass Without Coupling of Layers

Description

A simply-supported double-pane glass plate with a foil between both glass panes is subjected to a uniform pressure p. Considering only the small deformations and neglecting plate's self-weight, determine its maximum displacement $u_{z,max}$, in-plane stresses and stress ratios.

Material	Glass	Modulus of Elasticity	E_1	70000000.000	kPa
		Poisson's Ratio	ν_1	0.230	–
	Foil	Modulus of Elasticity	E_2	3000.000	kPa
		Poisson's Ratio	ν_2	0.499	–
Geometry	Plate	Side Length	L	10000.000	mm
		Top Layer Thickness	t_1	10.000	mm
		Middle Layer Thickness	t_2	0.380	mm
		Bottom Layer Thickness	t_3	15.000	mm
		Total Thickness	$t = \sum_{i=1}^{3} t_i$	25.380	mm
Load		Pressure	p	0.001	kPa

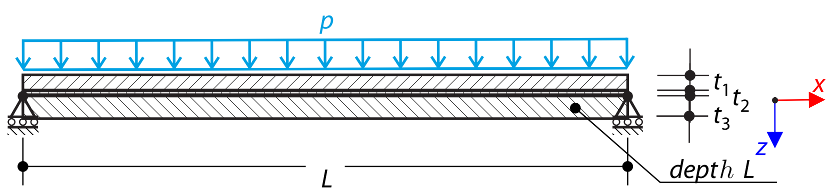

Figure 1: Problem sketch

Analytical Solution

At first, let us briefly explain the problematic of glass-foil-glass simulations. The foil is normally much softer and thinner than glass. Therefore the foil shear modulus G_2 is lower than the shear modulus of glass by 5-7 orders. In this case, normal vectors after deformation in glass and foil have significantly different directions. As a consequence, the usual 2D theories give unreliable results (too conservative results = underestimated displacements). The correct results can be obtained by 3D simulation, although these calculations are more time consuming. The displacement upper bound can be obtained by the 2D plate theory without layer's coupling, described in this example and available in the RF-GLASS module. All three possibilities are shown in **Figure 2**:

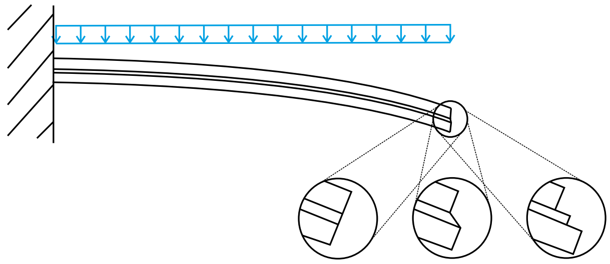

Figure 2: Left: 2D plate theories (fully coupled), Middle: 3D simulations (correct), Right: plate theories without layer's coupling

Although the plate setting is unsymmetrical, the unsymmetrical stiffness terms D_{16}, D_{17}, D_{18}, D_{27}, D_{28}, D_{38} are not present in this theory, because all panes are supposed to be located symmetrically on the plate middle plane (they are supposed to act independently). As a consequence, the elasticity matrix $\boldsymbol{D}$ is given as a sum of parts correspondent to individual layers ($\boldsymbol{D} = \sum_{i=1}^{n} \boldsymbol{D}_i$), i.e., the Steiner's part is not considered. In this verification example all the analytical results apply to the Kirchhoff plate theory, therefore in RFEM the same setting is considered.

The in-plane stiffness matrices, corresponding to the i-th layer, are as follows:

$$\boldsymbol{d}_i = \begin{bmatrix} \frac{E_i}{1-\nu_i^2} & \frac{\nu_i E_i}{1-\nu_i^2} & 0 \\ & \frac{E_i}{1-\nu_i^2} & 0 \\ \text{sym.} & & G_i \end{bmatrix} \qquad (28-1)$$

where $G_i = \frac{E_i}{2(1+\nu_i)}$ is a shear modulus and $i = 1, 2, 3$ is the number of the layer. The stiffness matrices corresponding to the i-th layer are than as follows:

$$\boldsymbol{D}_i = \begin{bmatrix} \frac{t_i^3}{12} d_{i,11} & \frac{t_i^3}{12} d_{i,12} & 0 & 0 & 0 & 0 & 0 & 0 \\ & \frac{t_i^3}{12} d_{i,11} & 0 & 0 & 0 & 0 & 0 & 0 \\ & & \frac{t_i^3}{12} d_{i,33} & 0 & 0 & 0 & 0 & 0 \\ & & & \frac{5}{6} G_i t_i & 0 & 0 & 0 & 0 \\ & & & & \frac{5}{6} G_i t_i & 0 & 0 & 0 \\ & & & & & t_i d_{i,11} & t_i d_{i,12} & 0 \\ & & & & & & t_i d_{i,11} & 0 \\ \text{sym.} & & & & & & & t_i d_{i,33} \end{bmatrix} \qquad (28-2)$$

The total stiffness matrix $\boldsymbol{D}$ is than defined by:

$$\boldsymbol{D} = \sum_{i=1}^{n} \boldsymbol{D}_i \qquad (28-3)$$

For the total stiffness matrix $\boldsymbol{D}$ the following relation holds:

$$\boldsymbol{f} = \boldsymbol{D}\boldsymbol{\varepsilon} \qquad (28-4)$$

where $\boldsymbol{f}$ is the stress resultant vector:

$$\boldsymbol{f} = \begin{bmatrix} m_x & m_y & m_{xy} & v_x & v_y & n_x & n_y & n_{xy} \end{bmatrix}^{\mathsf{T}} \tag{28 – 5}$$

and ε is the strain resultant vector:

$$\varepsilon = \begin{bmatrix} \kappa_x & \kappa_y & \kappa_{xy} & \gamma_x & \gamma_y & \varepsilon & \varepsilon_y & \varepsilon_{xy} \end{bmatrix}^{\mathsf{T}} \tag{28 – 6}$$

The stress vector resultant $\boldsymbol{f}$ is divided to stress resultants $\boldsymbol{f}_i$ corresponding to each layer according to the formula:

$$\boldsymbol{f} = \sum_{i=1}^{n} \boldsymbol{f}_i \tag{28 – 7}$$

where forces $\boldsymbol{f}_i$ are given by:

$$\boldsymbol{f}_i = \boldsymbol{D}_i \boldsymbol{D}^{-1} \boldsymbol{f} \tag{28 – 8}$$

To determine the maximum displacement $u_{z,max}$, the equivalent plate having the same properties as the combination of all three layers given above will be computed. The equivalent plate bending stiffness K_{eqv} is given by:

$$K_{\mathrm{eqv}} = \sum_{i=1}^{3} \frac{t_i^3}{12} \frac{E_i}{1 - \nu_i^2} \tag{28 – 9}$$

The maximum displacement can be than obtained by the following formula:

$$u_{z,\max} = \frac{16p}{k_{\mathrm{eqv}} \pi^6} \sum_{m,n=1} \frac{(-1)^{\frac{m+n-2}{2}}}{mn \left(\frac{m^2}{a^2} + \frac{n^2}{b^2} \right)^2} \tag{28 – 10}$$

To determine in-plane stresses, explicit formulae will be derived. These formulae determine the stress extreme values in the one-layered isotropic elastic Kirchhoff plate and can be expressed as:

$$\sigma_x \left(\frac{L}{2}, \frac{L}{2}, z \right) = \frac{192zp}{t^3 \pi^4} \sum_{m,n=1} (-1)^{\frac{m+n-2}{2}} \frac{\frac{m^2}{a^2} + \nu \frac{n^2}{b^2}}{mn \left(\frac{m^2}{a^2} + \nu \frac{n^2}{b^2} \right)^2} \tag{28 – 11}$$

$$\sigma_y \left(\frac{L}{2}, \frac{L}{2}, z \right) = \frac{192zp}{t^3 \pi^4} \sum_{m,n=1} (-1)^{\frac{m+n-2}{2}} \frac{\nu \frac{m^2}{a^2} + \frac{n^2}{b^2}}{mn \left(\frac{m^2}{a^2} + \nu \frac{n^2}{b^2} \right)^2} \tag{28 – 12}$$

$$\tau_{xy} (0, 0, z) = -\frac{192zp(1 - \nu)}{abt^3 \pi^4} \sum_{m,n=1} \frac{1}{mn \left(\frac{m^2}{a^2} + \nu \frac{n^2}{b^2} \right)^2} \tag{28 – 13}$$

where $a = b = L$. The moment resultants for the whole plate are given by formulae:

$$m_x\left(\frac{L}{2},\frac{L}{2}\right) = \sum_{m,n=1} \frac{16p}{mn\pi^2}(-1)^{\frac{m+n-2}{2}} \frac{\alpha_m^2 D_{11} + \beta_n^2 D_{12}}{D_{11}\alpha_m^4 + 2(D_{12}+2D_{33})\alpha_m^2\beta_n^2 + D_{22}\beta_n^4} \quad (28-14)$$

$$m_y\left(\frac{L}{2},\frac{L}{2}\right) = \sum_{m,n=1} \frac{16p}{mn\pi^2}(-1)^{\frac{m+n-2}{2}} \frac{\alpha_m^2 D_{11} + \beta_n^2 D_{12}}{D_{11}\alpha_m^4 + 2(D_{12}+2D_{33})\alpha_m^2\beta_n^2 + D_{22}\beta_n^4} \quad (28-15)$$

$$m_{xy}(0,0) = \sum_{m,n=1} \frac{16p}{mn\pi^2} \frac{-2D_{33}\alpha_m\beta_n}{D_{11}\alpha_m^4 + 2(D_{12}+2D_{33})\alpha_m^2\beta_n^2 + D_{22}\beta_n^4} \quad (28-16)$$

where $\alpha_m = \frac{m}{a}$ and $\beta_n = \frac{n}{b}$. To determine in-plane stresses, formulae relating maximum stresses and moment resultants must be derived. Maximum stress values at any point (x, y) can be expressed by following formulae:

$$\sigma_x(x,z,y) = \frac{2z}{t}\sigma_{x,\max}(x,y) \quad (28-17)$$

$$\sigma_y(x,z,y) = \frac{2z}{t}\sigma_{y,\max}(x,y) \quad (28-18)$$

$$\tau_{xy}(x,z,y) = \frac{2z}{t}\tau_{xy,\max}(x,y) \quad (28-19)$$

From these formulae, moments can be obtained by multiplying by z and integration over thickness:

$$m_x(x,y) = \int_{-t/2}^{t/2} \sigma_x(x,y,z)z\,\mathrm{d}z = \frac{t^2}{6}\sigma_{x,\max}(x,y) \quad (28-20)$$

$$m_y(x,y) = \int_{-t/2}^{t/2} \sigma_y(x,y,z)z\,\mathrm{d}z = \frac{t^2}{6}\sigma_{y,\max}(x,y) \quad (28-21)$$

$$m_{xy}(x,y) = \int_{-t/2}^{t/2} \tau_{xy}(x,y,z)z\,\mathrm{d}z = \frac{t^2}{6}\tau_{xy,\max}(x,y) \quad (28-22)$$

which yields for the i-th layer:

$$\sigma_{i,x,\max}(x,y) = \frac{6}{t^2}m_{i,x}(x,y) \quad (28-23)$$

$$\sigma_{i,y,\max}(x,y) = \frac{6}{t^2}m_{i,y}(x,y) \quad (28-24)$$

$$\tau_{i,xy,\max}(x,y) = \frac{6}{t^2}m_{i,xy}(x,y) \quad (28-25)$$

To determine stress ratios $\frac{\sigma_{1,x,\max}}{\sigma_{3,x,\max}}$ and $\frac{\tau_{1,xz,\max}}{\tau_{3,xz,\max}}$, another equation dependent on the z-coordinate has to be mentioned:

$$\tau_{i,xz}(z) = \tau_{i,xz,\max}\left(1 - \frac{4z^2}{t_i^2}\right) \quad (28-26)$$

It should be also mentioned that $z = 0$ at each pane's middle (as stated before, the panes are supposed to act independently). Because $t_2 \lll \min(t_1, t_3)$, the inner layer can be neglected in the following calculations. It is also supposed, that curvatures are the same in each pane, therefore:

$$\boldsymbol{m}_i = \frac{t_i^3}{12}\boldsymbol{d}_i\boldsymbol{\kappa} \quad (28-27)$$

where $\boldsymbol{d}_i$ is given by the formula **(28 – 1)**, $\boldsymbol{\kappa}$ is the curvature vector:

$$\boldsymbol{\kappa} = \begin{bmatrix} \kappa_x & \kappa_y & \kappa_{xy} \end{bmatrix}^{\mathrm{T}} \quad (28-28)$$

and $\boldsymbol{m}_i$ is the moment vector in the i-th pane, when $i = 1, 3$. At the same time, these vectors can be expressed as:

$$\boldsymbol{m}_i = \int_{-t_i/2}^{t_i/2} z\begin{bmatrix} \sigma_x & \sigma_y & \tau_{xy} \end{bmatrix}^{\mathrm{T}} \mathrm{d}z = \begin{bmatrix} \sigma_x & \sigma_y & \tau_{xy} \end{bmatrix}^{\mathrm{T}} \frac{t_i^6}{6} \quad (28-29)$$

Combining equations **(28 – 1)**, **(28 – 27)**, **(28 – 28)** and **(28 – 29)**, the desired ratio for the maximum normal stresses can be obtained:

$$\frac{\sigma_{1,x,max}}{\sigma_{3,x,max}} = \frac{t_1}{t_3} \quad (28-30)$$

The transversal shear ratios can be obtained from the transversal shear forces $v_{x,i}$ in x-direction:

$$v_{x,i} = \frac{5}{6}Gt_i\gamma_{xz} \quad (28-31)$$

where $i = 1, 3$. At the same time, these forces can be expressed as:

$$v_{x,i} = \int_{-t_i/2}^{t_i/2} \tau_{i,z}\mathrm{d}z = \frac{2}{3}\tau_{i,xz,\max,}t_i \quad (28-32)$$

Combining these two equations, the ratio for the maximum shear stresses can be obtained:

$$\frac{\tau_{1,xz,\max}}{\tau_{3,xz,\max}} = 1 \quad (28-33)$$

RFEM 5 Settings

- Modeled in version RFEM 5.03.0050
- The element size is $l_{FE} = 0.250$ m
- Geometrically linear analysis is considered
- The number of increments is 1
- The element type is plate
- The Mindlin plate theory is used
- Isotropic linear elastic material model is used
- Coupling of layers is not considered

Results

Structure File	Program
0028.01	RF-LAMINATE
0028.02	RF-GLASS

As can be seen below, excellent agreements of analytical results with numerical outputs were achieved:

Displacement	Analytical Solution	RF-LAMINATE		RF-GLASS	
	[mm]	[mm]	Ratio [-]	[mm]	Ratio [-]
$u_{z,\max}$	1.507	1.507	1.000	1.507	1.000

Stresses	Analytical Solution	RF-LAMINATE		RF-GLASS	
	[kPa]	[kPa]	Ratio [-]	[kPa]	Ratio [-]
$\sigma_{1,x,\max}\left(\frac{L}{2},\frac{L}{2}\right)$	62.160	62.112	0.999	62.112	0.999
$\sigma_{3,x,\max}\left(\frac{L}{2},\frac{L}{2}\right)$	93.200	93.168	1.000	93.168	1.000
$\tau_{1,xy,\max}(0,0)$	−49.020	−48.987	0.999	−48.987	0.999
$\tau_{3,xy,\max}(0,0)$	−73.493	−73.481	1.000	−73.481	1.000

Stress Ratios	Analytical Solution	RF-LAMINATE		RF-GLASS	
	[−]	[−]	Ratio [-]	[−]	Ratio [-]
$\frac{\sigma_{1,x,\max}}{\sigma_{3,x,\max}}$	0.667	0.667	1.000	0.667	1.000
$\frac{\tau_{1,xy,\max}}{\tau_{3,xy,\max}}$	1.000	1.000	1.000	1.000	1.000