Program: RFEM 5, RF-FE-LTB, RFEM 6

Category: Geometrically Linear Analysis, Second-Order Analysis, Isotropic Linear Elasticity, Warping, Member, Plate

Verification Example: 0053 – Cantilever Under the Torsion with Warping

0053 – Cantilever Under the Torsion with Warping

Description

A cantilever of I-profile is supported on the left end ($x = 0$) and it is loaded by the torque M according to the **Figure 1**. The aim of this example is to compare the fixed support with the fork support and to investigate the behaviour of some representative quantities. The comparison with the solution by means of plates is also made. The problem is described by the following set of parameters.

Material	Steel	Modulus of Elasticity	E	210000.000	MPa
		Shear Modulus	G	81000.000	MPa
Geometry	Cantilever	Length	L	5.000	m
		Height	h	400.000	mm
		Width	b	180.000	mm
		Web Thickness	s	10.000	mm
		Flange Thickness	t	14.000	mm
Load		Torque	M	1.000	kNm

Small deformations are considered and the self-weight is neglected. Determine the rotation in the half of the cantilever $\varphi_x(L/2)$ and in case of the member entity with warping determine the values of the primary torsional moment M_{Tpri}, the secondary torsional moment M_{Tsec} and the warping moment M_ω both on the left end (point A) and on the right end (point B).

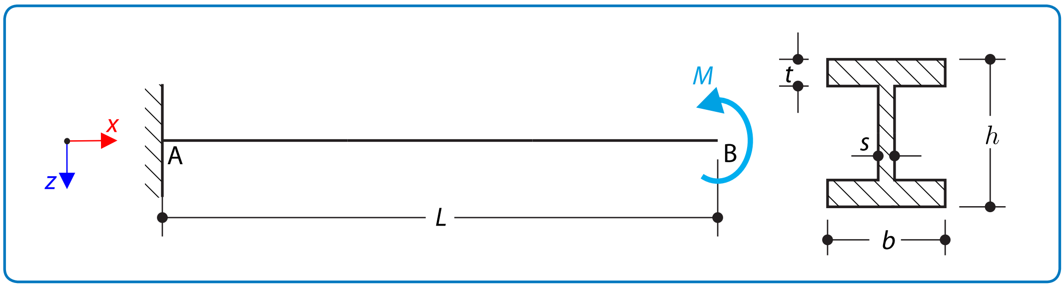

Figure 1: Problem sketch

Analytical Solution

When considering member calculation with warping the total torsional moment M_{T} is divided between the the primary torsional moment M_{Tpri} and the secondary torsional moment M_{Tsec} [1].

$$M_{\text{T}}(x) = M_{\text{Tpri}}(x) + M_{\text{Tsec}}(x) = GJ\varphi'(x) - EC_\omega\varphi'''(x) \tag{53 – 1}$$

where J is the torsional constant[1], C_ω is the warping constant[2], $\varphi(x) = \varphi_x$ and the dash denotes the differentiation with respect to the x. These constants are taken from RFEM 5 / RSTAB 8.

$$J = 441.813 \cdot 10^3 \text{ mm}^4 \tag{53 – 2}$$

$$C_\omega = 5.069 \cdot 10^{11} \text{ mm}^6 \tag{53 – 3}$$

The equation **(53 – 1)** can be expressed as follows.

$$\varphi'''(x) - \alpha^2 \varphi'(x) = -\frac{\alpha^2}{GJ} M_\mathrm{T}(x) \tag{53 – 4}$$

where α^2 is the constant defined as

$$\alpha^2 = \frac{GJ}{EC_\omega} \tag{53 – 5}$$

The equation **(53 – 4)** is the complete differential equation of the torsion. It can be solved by the method of initial parameters and results into following equations for the rotation $\varphi(x)$, relative twist $\varphi'(x)$ and warping moment M_ω.

$$\varphi(x) = \varphi(0) + \frac{\varphi'(0)}{\alpha} \sinh(\alpha x) - \frac{M_\omega(0)}{GJ} (\cosh(\alpha x) - 1) - \frac{M_\mathrm{T}(0)}{\alpha GJ} (\sinh(\alpha x) - \alpha x) \tag{53 – 6}$$

$$\varphi'(x) = \varphi'(0) \cosh(\alpha x) - \frac{M_\omega(0)}{GJ} \alpha \sinh(\alpha x) - \frac{M_\mathrm{T}(0)}{GJ} (\cosh(\alpha x) - 1) \tag{53 – 7}$$

$$M_\omega(x) = -EC_\omega \varphi''(x) = -\varphi'(0) \frac{GJ}{\alpha} \sinh(\alpha x) + M_\omega(0) \cosh(\alpha x) + \frac{M_\mathrm{T}(0)}{\alpha} \sinh(\alpha x) \tag{53 – 8}$$

Primary torsional moment M_Tpri and the secondary torsional moment M_Tsec can be then calculated.

$$M_\mathrm{Tpri} = GJ\varphi'(x) = \varphi'(0) GJ \cosh(\alpha x) - M_\omega(0) \alpha \sinh(\alpha x) - M_\mathrm{T}(0) (\cosh(\alpha x) - 1) \tag{53 – 9}$$

$$M_\mathrm{Tsec} = \frac{\mathrm{d}M_\omega(x)}{\mathrm{d}x} = -\varphi'(0) GJ \cosh(\alpha x) + M_\omega(0) \alpha \sinh(\alpha x) + M_\mathrm{T}(0) \cosh(\alpha x) \tag{53 – 10}$$

The warping is restrained, when the fixed support is considered. Boundary conditions in this case are following.

[1] The torsional constant J can be also denoted as I_T.

[2] The warping constant C_ω can be also denoted as I_ω or J_ω. It is defined as $C_\omega = \int_A \omega^2 \mathrm{d}A$, where ω is the warping unit [1].

$$\varphi(0) = 0$$
$$\varphi'(0) = 0$$
$$M_T(0) = M$$
$$M_\omega(L) = 0$$

From those conditions the warping moment on the fixed end $M_\omega(0)$ can be calculated.

$$M_\omega(0) = -\frac{M \sinh(\alpha L)}{\alpha \cosh(\alpha L)} \tag{53 – 11}$$

Using this expression and other mentioned boundary conditions desired quantities can be calculated. When the fork support is considered, the warping is enabled and the boundary conditions are following.

$$\varphi(0) = 0$$
$$M_\omega(0) = 0$$
$$M_T(0) = M$$
$$M_\omega(L) = 0$$

Using those boundary conditions the rotation $\varphi(x)$ results into well-known formula.

$$\varphi(x) = \frac{Mx}{GJ} \tag{53 – 12}$$

The warping moment M_ω and the secondary torsional moment M_{Tsec} then equal to the zero.

RFEM 5 Settings

- Modeled in RFEM 5.05.0029 and RFEM 6.01
- The element size is $l_{FE} = 0.025$ m
- Isotropic linear elastic material model is used
- Kirchhoff plate bending theory is used
- Torsional Warping and Steel Design add-on is used in RFEM 6

Results

Structure File	Program	Entity	Support
0053.01	RFEM 5 – RF-FE-LTB, RFEM 6	Member	Fixed Support
0053.02	RFEM 5 – RF-FE-LTB, RFEM 6	Member	Fork Support
0053.03	RFEM 5, RFEM 6	Plate	Fixed Support
0053.04	RFEM 5, RFEM 6	Plate	Fork Support

In the following graphs there are shown behaviours of the total torsional moment M_T, the primary torsional moment M_{Tpri}, the secondary torsional moment M_{Tsec} and the warping moment M_ω calculated in RFEM 5, RF-FE-LTB module, when the fixed support is considered. Values on the both ends of the cantilever are compared with the analytical solution and solution in ANSYS 13.

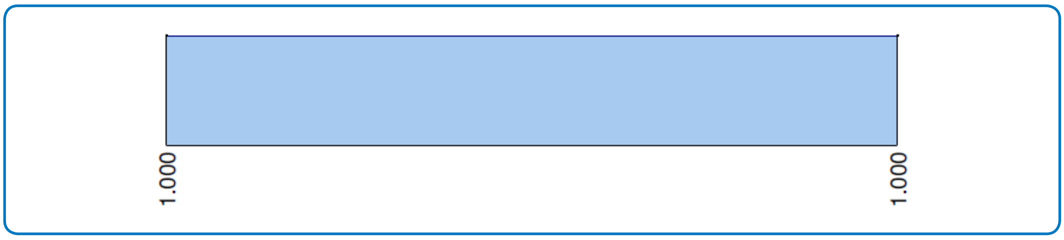

Figure 2: Total torsional moment M_T [kNm] behaviour

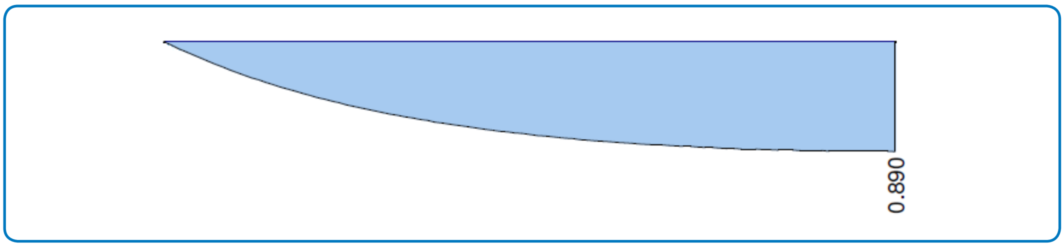

Figure 3: Primary torsional moment M_{Tpri} [kNm] behaviour

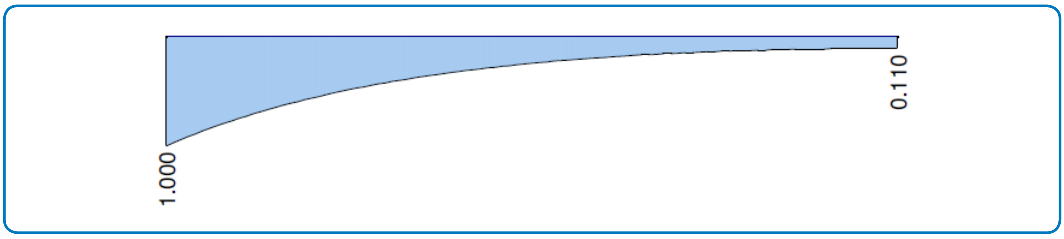

Figure 4: Secondary torsional moment M_{Tsec} [kNm] behaviour

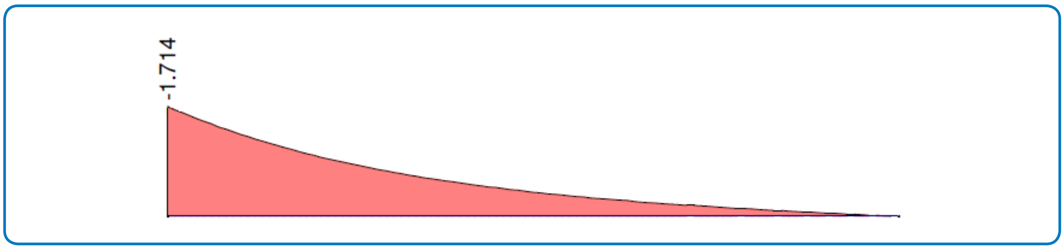

Figure 5: Warping moment M_ω [kNm^2] behaviour

Point A ($x = 0$)	Analytical Solution	ANSYS 13 *	Ratio [-]	RFEM 5 RF-FE-LTB	Ratio	RFEM 6	Ratio [-]
M_{Tpri} [kNm]	0.000	0.008	-	0.000	-	0.000	-
M_{Tsec} [kNm]	1.000	0.992	0.992	1.000	1.000	1.000	1.000
M_ω [kNm^2]	-1.714	-1.683	0.989	-1.714	1.000	-1.743	1.017

Point B ($x = L$)	Analytical Solution	ANSYS 13 *	Ratio [-]	RFEM 5 RF-FE-LTB	Ratio	RFEM 6	Ratio [-]
M_{Tpri} [kNm]	0.890	0.893	1.003	0.890	1.000	0.869	0.976
M_{Tsec} [kNm]	0.110	0.107	0.973	0.110	1.000	0.131	1.191
M_ω [kNm^2]	0.000	0.001	-	0.000	-	0.000	-

The calculated rotation round the x-axis can be compared with the results, when the fork support is considered and also with plate models (**Figure 6**), which take the warping naturally into the account. The rotation at the half length $\varphi_x(L/2)$ is used due to the affected area in the nearby of the loading point in case of plate models. The definition of the fork support in case of the plate model is complicated, because the warping moment has to equal to the zero (i. e. $\varphi''(0) = 0$). This boundary condition can not be set directly in RFEM 5 / RFEM 6. The approximation shown in **Figure 7** is used. The rotation round the x-axis is restrained on all lines of the support. Only the central node (on the axis) is fully fixed. In case of fixed support all the lines of the support are fixed (**Figure 7**).

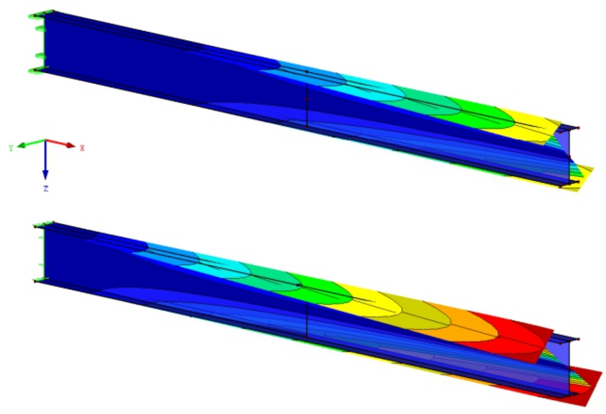

Figure 6: Cantilever with fixed support (top) and fork support (bottom) modeled by means of plates

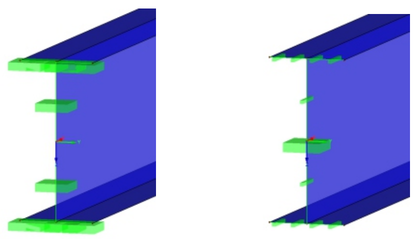

Figure 7: Fixed support and approximation of the fork support in RFEM 5 / RFEM 6

Support	Analytical Solution	ANSYS 13 *		RFEM 5, RF-FE-LTB (Member)		RFEM 5 (Plate)	
	$\varphi_x(L/2)$ [mrad]	$\varphi_x(L/2)$ [mrad]	Ratio [-]	$\varphi_x(L/2)$ [mrad]	Ratio [-]	$\varphi_x(L/2)$ [mrad]	Ratio [-]
Fixed Support	32.6	32.2	0.988	32.6	1.000	32.5	0.997
Fork Support	69.9	68.5	0.979	69.9	1.000	68.1	0.974**

Support	Analytical Solution	ANSYS 13 *		RFEM 6 (Member)		RFEM 6 (Plate)	
	$\varphi_x(L/2)$ [mrad]	$\varphi_x(L/2)$ [mrad]	Ratio [-]	$\varphi_x(L/2)$ [mrad]	Ratio [-]	$\varphi_x(L/2)$ [mrad]	Ratio [-]
Fixed Support	32.6	32.2	0.988	32.4	0.994	32.7	1.003
Fork Support	69.9	68.5	0.979	69.9	1.000	68.8	0.999**

* Remark: Numerical solution in ANSYS 13 was carried out by the company Designtec s.r.o. The quantities M_{Tsec} and M_{Tpri} are not the original results from ANSYS 13. They are calculated from the warping moment M_{ω} by means of formulae **(53 – 10)** and **(53 – 1)**. Thus they should no not be taken as entirely accurate values. Elements BEAM188 are used.

** Remark: The solution with plate models is used as a demonstration of the warping effect. The relative error is caused also by the approximation of the fork support.

References

[1] JANATKA, J. *Přímé tenkostěnné nosníky (Teorie výpočtu)*. SNTL Praha, 1961.