0067 – Incompressible Material – Thick-Walled Vessel

Description

This example is a modification of Verification Example 0064 [1], where the only difference is that the material of the vessel is incompressible. A thick-walled vessel is loaded by inner and outer pressure. The vessel is open-ended, thus there is no axial stress. The problem is modeled as a quarter model (see **Figure 1**) and described by the following set of parameters.

Material	Modulus of Elasticity	E	1.000	MPa
	Poisson's Ratio	ν	0.499	–
Geometry	Inner Radius	r_1	200.000	mm
	Outer Radius	r_2	300.000	mm
Load	Inner Pressure	p_1	0.060	MPa
	Outer Pressure	p_2	0.001	MPa

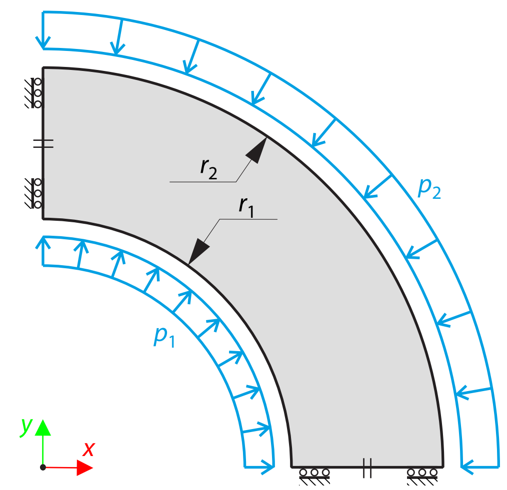

Figure 1: Problem Sketch

While neglecting self-weight, determine the radial deflection of the inner and outer radius $u_r(r_1)$, $u_r(r_2)$.

Volumetric Locking

The aim of this verification example is to show the phenomenon of volumetric locking of finite elements. This problem can occur in incompressible material cases when the Poisson ratio ν is approaching 0.5 (rubber materials, materials in plastic state). In this case, the finite element displacements tend to zero. The volumetric locking cannot be avoided by the mesh refinement. All fully integrated elements will lock when the incompressible material is used. The simplest way to avoid locking is using reduced integration. In this case the number of integration points is reduced. The integration scheme is one order less accurate than the standard integration scheme. In RFEM the reduced integration is used so the result will be correct, as seen bellow.

Analytical Solution

The analytical solution is the same as in the case of Verification Example 0064, see [1]. The stress state of the thick-walled vessel is described by the equation of equilibrium and corresponding boundary conditions

$$\frac{d\sigma_r}{dr} + \frac{\sigma_r - \sigma_t}{r} = 0 \tag{67 – 1}$$

$$\sigma_r(r_1) = -p_1 \tag{67 – 2}$$

$$\sigma_r(r_2) = -p_2 \tag{67 – 3}$$

where σ_r, σ_t and r is the radial stress, tangential stress and radius, respectively. Isotropic linear elasticity is considered, that is, the constitutive relation between stress and strain is described by Hooke's Law

$$\sigma_t(r) = \frac{E}{1-\nu^2}\left[\varepsilon_t(r) + \nu\varepsilon_r(r)\right] \tag{67 – 4}$$

$$\sigma_r(r) = \frac{E}{1-\nu^2}\left[\varepsilon_r(r) + \nu\varepsilon_t(r)\right] \tag{67 – 5}$$

where the radial and tangential strains – ε_r and ε_t – are coupled with the radial deflection u_r through

$$\varepsilon_r(r) = \frac{du_r(r)}{dr} \tag{67 – 6}$$

$$\varepsilon_t(r) = \frac{u_r(r)}{r} \tag{67 – 7}$$

Substituting **(67 – 4)**—**(67 – 7)** back into **(67 – 1)**, the following second-order differential equation is obtained

$$\frac{d^2u_r(r)}{dr^2} + \frac{du_r(r)}{dr} - \frac{u_r(r)}{r} = 0 \tag{67 – 8}$$

the solution of which is assumed to be in polynomial form

$$u_r(r) = r^n \tag{67 – 9}$$

Hence, **(67 – 8)** yields $n = \pm 1$, therefore the solution can be written as a linear combination

$$u_r(r) = C_1 r + \frac{C_2}{r} \tag{67 – 10}$$

which, combined with Hooke's Law **(67 – 4)**, **(67 – 5)**, leads to the expression for the corresponding stresses

$$\sigma_t(r) = K + \frac{C}{r^2} \quad (67-11)$$

$$\sigma_r(r) = K - \frac{C}{r^2} \quad (67-12)$$

where K, C are real constants obtained from exploiting the boundary conditions **(67 – 2)**, **(67 – 3)**

$$K = \frac{p_1 r_1^2 - p_2 r_2^2}{r_2^2 - r_1^2} \quad (67-13)$$

$$C = (p_1 - p_2)\frac{r_1^2 r_2^2}{r_2^2 - r_1^2} \quad (67-14)$$

The radial deflection of the inner and outer radius of the open-ended vessel $u_r(r_1)$, $u_r(r_2)$ is then determined by the substitution of the aforementioned relations into Hooke's Law

$$u_r(r_1) = \frac{r_1}{E}\left[\sigma_t(r_1) - \nu\sigma_r(r_1)\right] \approx 29.988 \text{ mm} \quad (67-15)$$

$$u_r(r_2) = \frac{r_2}{E}\left[\sigma_t(r_2) - \nu\sigma_r(r_2)\right] \approx 22.497 \text{ mm} \quad (67-16)$$

RFEM Settings

- Modeled in RFEM 5.07 and RFEM 6.01
- The element size is $l_{FE} = 0.002$ m
- Isotropic linear elastic material model is used

Results

Structure Files	Program
0067.01	RFEM 5, RFEM 6

Quantity	Analytical Solution	RFEM 5	Ratio	RFEM 6	Ratio
$u_r(r_1)$ [mm]	29.988	29.987	1.000	29.986	1.000
$u_r(r_2)$ [mm]	22.497	22.496	1.000	22.495	1.000

References

[1] DLUBAL SOFTWARE GMBH, *Verification Example 0064 – Thick-Walled Vessel*. 2016.