Verification Example

Program: RFEM 5, RF-DYNAM Pro

Category: Geometrically Linear Analysis, Isotropic Linear Elasticity, Dynamics, Member

Verification Example: 0101 – Modal Analysis of the Cantilever

0101 – Modal Analysis of the Cantilever

Description

A steel cantilever of rectangular cross-section is fully fixed on one side and free on the other side according to the **Figure 1**. The aim of this verification example is to determine the natural frequencies of the structure. The problem is described by the following parameters.

Material	Steel	Modulus of Elasticity	E	208000.000	MPa
		Poisson's Ratio	ν	0.300	–
		Density	ρ	7800.000	kg/m^3
Geometry		Length	L	90.000	mm
		Width	w	10.000	mm
		Thickness	t	5000	mm

Determine the natural frequencies of longitudinal and transversal oscillations.

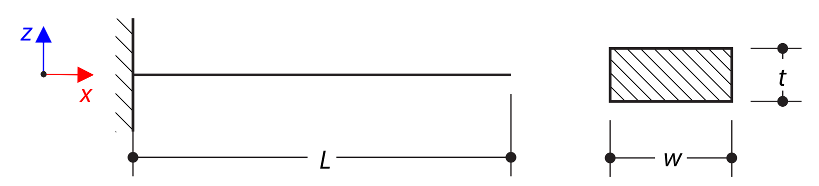

Figure 1: Problem sketch

Analytical Solution

The natural longitudinal oscillation of a thin bar is described by the following differential equation, [1].

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \qquad (101 – 1)$$

Where u denotes the deflection in the longitudinal direction and c is the material constant which is describing the longitudinal waves propagation velocity.

$$c = \sqrt{\frac{E}{\rho}} \qquad (101 – 2)$$

The solution $u(x, t)$ is supposed in the following form.

$$u(x,t) = U(x)T(t) \tag{101 – 3}$$

Using formula **(101 – 3)** and its derivative the formula **(101 – 1)** can be rewritten.

$$-\frac{1}{T}\frac{\partial^2 T}{\partial t^2} = c^2 \frac{1}{U}\frac{\partial^2 U}{\partial x^2} = \text{const.} = \omega^2 \tag{101 – 4}$$

Where ω is the angular frequency. For each time has to be fulfilled following equation.

$$\frac{\partial^2 U}{\partial x^2} + \frac{\omega^2}{c^2} U = 0 \tag{101 – 5}$$

This is the second order differential equation with the following solution.

$$U(x) = C_1 \sin\left(\frac{\omega^2}{c^2} x\right) + C_2 \cos\left(\frac{\omega^2}{c^2} x\right) \tag{101 – 6}$$

The constants C_1 and C_2 can be obtained from the boundary conditions. The deflection on the fixed side and the stress on the free side has to be zero[1].

$$U(0) = 0 \tag{101 – 7}$$

$$\frac{\partial U(L)}{\partial x} = 0 \tag{101 – 8}$$

Using the boundary conditions the wave equation is determined.

$$C_1 \frac{\omega}{c} \cos\left(\frac{\omega}{c} L\right) = 0 \tag{101 – 9}$$

The set of angular frequencies can be then obviously determined.

$$\omega_k = \frac{c}{L}(2k-1)\frac{\pi}{2}, \qquad k = 1, 2, 3, \ldots \tag{101 – 10}$$

The first natural frequency in longitudinal direction can be calculated.

$$f_1 = \frac{\omega_1}{2\pi} = \frac{c}{4L} = 14275.253 \text{ Hz} \tag{101 – 11}$$

The natural transversal oscillation of a thin bar is described by the following differential equation, [1].

$$\frac{\partial^4 v}{\partial x^4} + \frac{\rho A}{EI}\frac{\partial^2 v}{\partial t^2} = 0 \tag{101 – 12}$$

[1] The stress can be expressed as $\sigma = E\varepsilon = E\frac{\mathrm{d}U(x)}{\mathrm{d}x}$.

Where I is the quadratic moment of the cross-section and A is the cross-section area. Note that the equation **(101 – 12)** has to be solved for both transversal directions. The solution $v(x,t)$ is supposed in the following form.

$$v(x,t) = V(x)T(t) \tag{101 – 13}$$

Using formula **(101 – 3)** and its derivative the formula **(101 – 12)** can be rewritten.

$$-\frac{1}{T}\frac{\partial^2 T}{\partial t^2} = \frac{EI}{\rho A}\frac{1}{V}\frac{\partial^4 V}{\partial x^4} = \text{const.} = \omega^2 \tag{101 – 14}$$

Where ω is the angular frequency. For each time has to be fulfilled following equation.

$$\frac{\mathrm{d}^4 V}{\mathrm{d}x^4} - \lambda^4 V = 0 \tag{101 – 15}$$

Where λ^4 is the substitution for following expression.

$$\lambda^4 = \frac{\rho A \omega^2}{EI} \tag{101 – 16}$$

The fourth order differential equation has the following solution.

$$V(x) = C_1 \sinh(\lambda x) + C_2 \cosh(\lambda x) + C_3 \sin(\lambda x) + C_4 \cos(\lambda x) \tag{101 – 17}$$

The constants C_1 till C_4 can be obtained from the boundary conditions. The deflection and the rotation on the fixed side and the bending moment M and the transversal force T on the free side has to be zero. The bending moment can be defined by means of Bernoulli-Euler formula. The transversal force can be determine using Schwedler theorem.

$$V(0) = 0 \tag{101 – 18}$$

$$\frac{\mathrm{d}V(0)}{\mathrm{d}x} = 0 \tag{101 – 19}$$

$$M(L) = EI\frac{\mathrm{d}^2 V(L)}{\mathrm{d}x^2} = 0 \tag{101 – 20}$$

$$T(L) = EI\frac{\mathrm{d}^3 V(L)}{\mathrm{d}x^3} = 0 \tag{101 – 21}$$

Using the boundary conditions the wave equation for the transversal oscillations is determined.

$$\cosh(\lambda L)\cos(\lambda L) + 1 = 0 \tag{101 – 22}$$

This is the transcendent equation and its solution can be found numerically. The natural frequencies can be then determined.

$$f_i = \frac{1}{2\pi}\lambda_i^2\sqrt{\frac{EI}{\rho A}}, \qquad i = 1, 2, 3, \ldots \tag{101 – 23}$$

RFEM 5 Settings

- Modeled in RFEM 5.08.02
- The global element size is $l_{\mathrm{FE}} = 0.001$ m
- Shear stiffness of the members is neglected
- Isotropic linear elastic material model is used
- Member entity is used

Results

Structure Files	Program
0101.01	RF-DYNAM Pro

Natural Frequency (x - direction)	Analytical Solution	RF-DYNAM	Ratio
f_{x1} [Hz]	14275.253	14275.072	1.000

Natural Frequency (y - direction)	Analytical Solution	RF-DYNAM	Ratio
f_{y1} [Hz]	1024.900	1024.820	1.000
f_{y2} [Hz]	6422.940	6419.154	0.999
f_{y3} [Hz]	17984.417	17960.779	0.999

Natural Frequency (z - direction)	Analytical Solution	RF-DYNAM	Ratio
f_{z1} [Hz]	512.450	512.495	1.000
f_{z2} [Hz]	3211.470	3211.004	1.000
f_{z3} [Hz]	8992.208	8988.420	1.000

References

[1] STEJSKAL, V. and OKROUHLÍK, M. *Kmitání s Matlabem*. Vydavatelství ČVUT Praha.