0102 – Transient Response to a Constant Force

Description

A long thin beam is carrying a concentrated mass m and it is loaded by means of the time dependent force $F(t)$. It is simply supported according to the **Figure 1**. The problem is described by the following parameters.

Material	Steel	Modulus of Elasticity	E	200000.000	MPa
		Poisson's Ratio	ν	0.300	—
Geometry		Length	L	10.000	m
		Width and Height	h	0.050	m
Load		Concentrated Mass	m	50.000	kg
		Constant Force	F_0	1.000	kN
		Time	t_0	1.000	s

The self weight of the beam is neglected. Determine the deflections u_z in given test times.

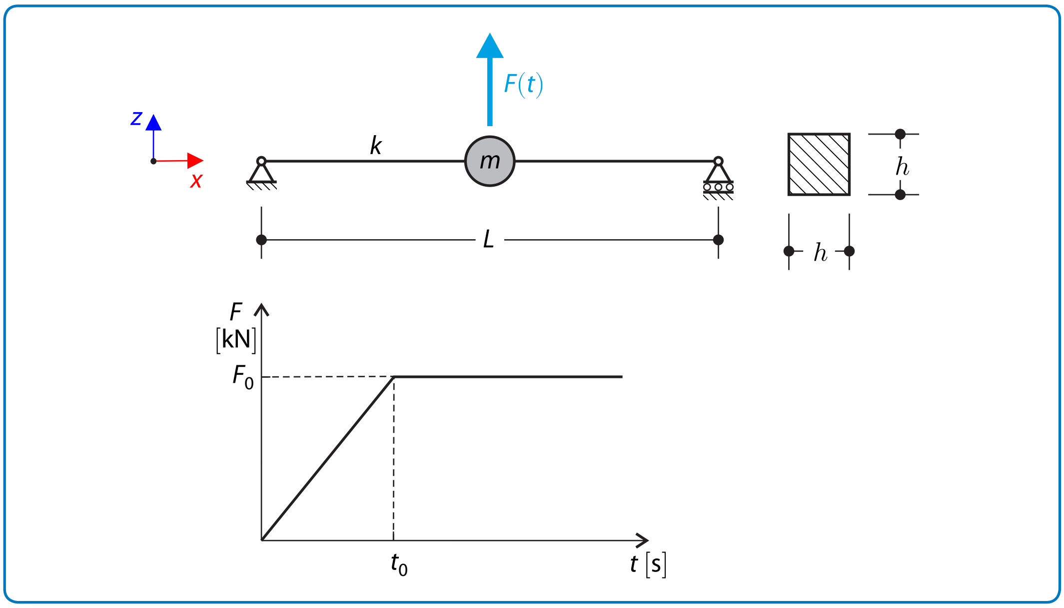

Figure 1: Problem sketch

Analytical Solution

At first, it is necessary to determine the bending stiffness of the beam, which behaves as a spring. The stiffness can be determined from the known formula for the maximum deflection of the simply supported beam.

$$u_{z,\max} = \frac{FL^3}{48EI_y} \tag{102 – 1}$$

Where I_y is the quadratic moment of the cross-section defined as $I_y = \frac{h^4}{12}$.The stiffness can be then determined from the following definition.

$$k = \frac{F}{u_{z,\max}} = \frac{48EI_y}{L^3} \tag{102 – 2}$$

Given problem is described by the second-order differential equation, [1].

$$m\frac{\mathrm{d}^2 u_z(t)}{\mathrm{d}t^2} + ku_z(t) = F(t) \tag{102 – 3}$$

The loading force $F(t)$ is defined differently during the loading.

$$0 \leq t \leq t_0: \quad F(t) = \frac{F_0}{t_0}t \tag{102 – 4}$$

$$t \geq t_0: \quad F(t) = F_0 \tag{102 – 5}$$

The differential equation **(102 – 3)** has the following solution.

$$u_z = A\sin(\Omega t) + B\cos(\Omega t) + \frac{F(t)}{k} \tag{102 – 6}$$

Where Ω is the natural angular frequency defined as $\Omega = \sqrt{k/m}$ and A and B are arbitrary constants. The equation **(102 – 6)** has to be solved for both time intervals separately because of the different definition of the loading force $F(t)$.

$$u_{z1} = C_1\sin(\Omega t) + C_2\cos(\Omega t) + \frac{F_0}{t_0 k}t \tag{102 – 7}$$

$$u_{z2} = C_3\sin(\Omega t) + C_4\cos(\Omega t) + \frac{F_0}{k} \tag{102 – 8}$$

C_1 till C_4 are arbitrary constants which can be determined from the following initial conditions.

$$u_{z1}(0) = 0$$

$$\frac{\mathrm{d}u_{z1}(0)}{\mathrm{d}t} = 0$$

$$u_{z1}(t_0) = u_{z2}(t_0)$$

$$\frac{\mathrm{d}u_{z1}(t_0)}{\mathrm{d}t} = \frac{\mathrm{d}u_{z2}(t_0)}{\mathrm{d}t}$$

$$C_1 = -\frac{F_0}{t_0 k}\sqrt{\frac{m}{k}}$$

$$C_2 = 0$$

$$C_3 = \frac{F_0\left(\cos(\Omega t_0) - 1\right)}{t_0 k \Omega}$$

$$C_4 = -\frac{F_0 \sin(\Omega t_0)}{t_0 k \Omega}$$

With the calculated constants the solution in both time intervals can be determined. The extremal points of the solution in the second time interval ($0 \leq t \leq t_0$) are given by the following formula.

$$t_i = \sqrt{\frac{m}{k}}\left(\arctan\left(\frac{1-\cos(\Omega t_0)}{\sin(\Omega t_0)}\right) + i\pi\right), \quad i = 4, 5, 6, \ldots \tag{102 – 9}$$

Note that the parameter i is chosen from the number four to get the extremal points in the second time interval. The deflections are compared in the first three extremal points which are taken as test times.

RFEM 5 Settings

- Modeled in RFEM 5.07.03.118540
- The global element size is $l_{\mathrm{FE}} = 0.001$ m
- Isotropic linear elastic material model is used
- Member entity is used
- Shear stiffness of the members is neglected

Results

Structure Files	Program
0102.01	RF-DYNAM Pro

Test time	Analytical Solution	RF-DYNAM Pro Solution	
	u_z [m]	u_z [mm]	Ratio [-]
$t_1 = 1.128$ s	238.360	238.361	1.000
$t_2 = 1.442$ s	161.643	161.645	1.000
$t_3 = 1.757$ s	238.360	238.361	1.000

References

[1] STEJSKAL, V. and OKROUHLÍK, M. *Kmitání s Matlabem*. Vydavatelství ČVUT Praha.