0105 – Equivalent Loads

Description

A cantilever beam with I-beam cross-section of length L is defined. The beam has five mass points with masses m acting in X-direction. The self-weight is neglected. The frequencies, the mode shapes and the equivalent loads of this 5-DOF system are analytically calculated and compared with results from RSTAB and RFEM.

Material	Isotropic Linear Elastic	Modulus of Elasticity	E	2.1×10^{11}	Pa
		Shear Modulus	G	8.1×10^{10}	Pa
Structure	Cantilever	Length	L	5.0	m
	Cross Section IPE 300	Depth	d	300.0	mm
		Width	b	150.0	mm
		Web thickness	t_w	7.1	mm
		Flange thickness	t_f	10.7	mm
		Radius	r	15.0	mm
		Area	A	5.381×10^{-3}	m^2
		Moment of Inertia	I_y	8.356×10^{-5}	m^4
			I_z	6.038×10^{-6}	m^4
5-DOF System	Nodal Mass		m	1000.0	kg
	Distance Nodes		L_N	1.0	m

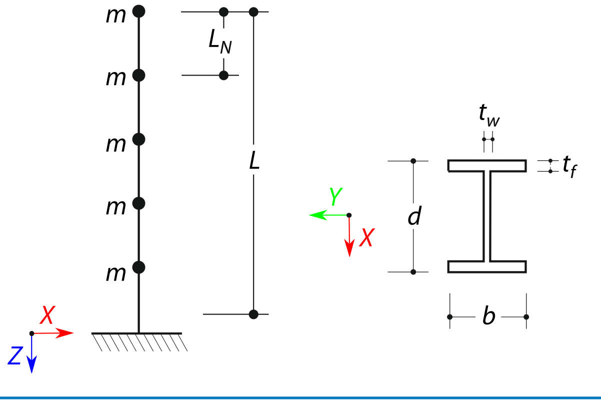

Figure 1: Problem sketch

Analytical Solution

Eigenvalue Analysis of the cantilever beam

The equation of motion for a MDOF system is defined with

$$\mathbf{M}\,\ddot{\mathbf{u}} + \mathbf{K}\,\mathbf{u} = \mathbf{0} \qquad (105-1)$$

where **K** is the stiffness matrix which can also be expressed via the flexibility matrix with $\mathbf{F} = \mathbf{K}^{-1}$, **M** is the mass matrix, and **ü** and **u** are the acceleration and displacement vectors, respectively.

The flexibility matrix **F** of the cantilever beam is determined, neglecting the shear stiffness of the member:

$$\mathbf{F} = \frac{1}{EI_y}\begin{bmatrix} 125/3 & 88/3 & 18 & 26/3 & 7/3 \\ & 64/3 & 27/2 & 20/3 & 11/6 \\ & & 9 & 14/3 & 4/3 \\ & & & 8/3 & 5/6 \\ & & & & 1/3 \end{bmatrix} \qquad (105-2)$$

The stiffness matrix **K** is the inverse of the flexibility matrix **F** and is given as follows:

$$\mathbf{K} = EI_y\begin{bmatrix} 291/181 & -660/181 & 468/181 & -126/181 & 36/181 \\ & 1788/181 & -1722/181 & 756/181 & -216/181 \\ & & 2544/181 & -1938/181 & 864/181 \\ & & & 2652/181 & -2154/181 \\ & & & & 3408/181 \end{bmatrix} \qquad (105-3)$$

The mass matrix for this structure is a simple diagonal matrix. The masses m are lumped together in 5 nodes acting in X-direction only.

$$\mathbf{M} = \begin{bmatrix} m & 0 & 0 & 0 & 0 \\ 0 & m & 0 & 0 & 0 \\ 0 & 0 & m & 0 & 0 \\ 0 & 0 & 0 & m & 0 \\ 0 & 0 & 0 & 0 & m \end{bmatrix} \qquad (105-4)$$

To determine the frequencies of the system the following eigenvalue problem is solved.

$$[\mathbf{F}\,\mathbf{M}\,\boldsymbol{\lambda} + \mathbf{I}]\;\mathbf{u} = \mathbf{0} \qquad (105-5)$$

where $\boldsymbol{\lambda}$ are the eigenvalues that correspond to the angular frequencies $\omega = \sqrt{\boldsymbol{\lambda}}$, and **u** are the mode shapes. To determine the eigenvalues λ the determinant of the eigenvalue problem has to be assembled and the resulting polynomial of 5^{th} order has to be solved. Once $\boldsymbol{\lambda}$ is known, the mode shapes **u** can be determined. For further details the reader is referred to the literature, i.e. [1–3]. The resulting angular frequencies ω are listed below, for simplicity they are written as a row vector with the columns representing each mode i:

$$\omega = \begin{bmatrix} 15.5145 & 99.0647 & 280.6927 & 542.4441 & 807.8413 \end{bmatrix} \quad (105-6)$$

The natural periods **T** are listed as they are required later for the calculation of equivalent loads.

$$\mathbf{T} = \begin{bmatrix} 0.4050 & 0.0634 & 0.0224 & 0.0116 & 0.0078 \end{bmatrix} \quad (105-7)$$

The masses are only acting in X-direction, so the mode shapes $\mathbf{u}$ contain only $\mathbf{u_X}$. The mode shapes are scaled so that $M_i = \mathbf{u}_i^T\,\mathbf{M}\,\mathbf{u}_i = 1$ kg, where M_i are the modal masses for each eigenvalue i. The corresponding displacements at each node (rows in matrix) and for each mode (columns in matrix) are as follows:

$$\mathbf{u} = \begin{bmatrix} 0.0238 & 0.0164 & -0.0109 & -0.0063 & 0.0026 \\ 0.0171 & -0.0053 & 0.0168 & 0.0175 & -0.0096 \\ 0.0107 & -0.0183 & 0.0081 & -0.0147 & 0.0164 \\ 0.0053 & -0.0177 & -0.0172 & -0.0042 & -0.0186 \\ 0.0015 & -0.0072 & -0.0154 & 0.0205 & 0.0170 \end{bmatrix} \quad (105-8)$$

The participation factors Γ_i are required for each mode i for the calculation of equivalent loads, they are defined as follows:

$$\Gamma_i = \frac{1}{M_i}\,\mathbf{u}_i^T\,\mathbf{M} \quad (105-9)$$

The participation factors for each mode of the cantilever beam are calculated.

$$\boldsymbol{\Gamma} = \begin{bmatrix} 58.2542 & -32.1192 & -18.7208 & 12.8335 & 7.7225 \end{bmatrix} \quad (105-10)$$

Response Spectrum

The user-defined response spectrum that is used to determine the equivalent loads is shown in **Figure 2**.

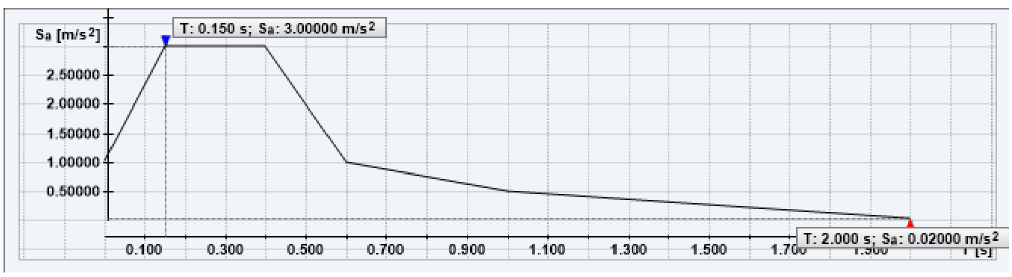

Figure 2: Response spectrum: Acceleration S_a $\mathrm{m/s^2}$ versus the natural period T.

For each natural period T_i (see **Equation 7**) the values of S_a are determined:

$$\mathbf{S_a} = \begin{bmatrix} 2.9500 & 1.8459 & 1.2987 & 1.1547 & 1.1039 \end{bmatrix} \quad (105-11)$$

Equivalent Loads

The definition of equivalent loads (see RF-DYNAM Pro manual [4]) simplify for this example. The masses are only acting in X-direction and the structure is only excited in the X-direction. The equivalent loads on each node separate for each mode i is defined with:

$$\mathbf{F}_i = \mathbf{M} \cdot \Gamma_i \cdot \mathbf{u}_i \cdot S_a(T_i) \quad (105-12)$$

with

$\mathbf{M}$ Mass matrix as defined in **Equation 4**

Γ_i Participation factors as given in **Equation 10**

$\mathbf{u_i}$ Displacement values of the mode shapes scaled as given in **Equation 8**

$S_a(T)$ Acceleration from the response spectra corresponding to the natural period T of the mode i, see **Equation 11**

The resulting equivalent loads **F** at each node (rows in the matrix) and for each mode i (columns in the matrix) are determined:

$$\mathbf{F} = \begin{bmatrix} 4083.0409 & -975.0779 & 265.1163 & -92.7467 & 22.4528 \\ 2930.4956 & 314.3735 & -407.9304 & 259.5644 & -82.1082 \\ 1840.6565 & 1087.7982 & -195.8764 & -217.9269 & 139.4213 \\ 0907.2363 & 1047.7774 & 418.7320 & -062.4139 & -158.4469 \\ 0249.5531 & 429.44060 & 375.1103 & 303.6996 & 144.5150 \end{bmatrix} \quad (105-13)$$

Results

In RSTAB DYNAM Pro and RFEM RF-DYNAM Pro, with the add-on module Equivalent Loads, a multi-modal response spectra can be performed where equivalent loads separate for each eigenvalue and separate for each excitation direction are exported into Load Cases (LC). The LCs are calculated in RFEM / RSTAB and internal forces result.

RFEM 5 and RSTAB 8 Settings

- Modelled in version RFEM 5.05.0029 and RSTAB 8.05.0029
- The members are not divided into finite elements (RFEM) nor internal nodes (RSTAB)
- Isotropic linear elastic material model is used
- Shear stiffness of members is deactivated
- Linear dynamic analysis is performed
- The Root of the characteristic polynomial is used as eigenvalue solver in RFEM RF-DYNAM Pro
- Subspace iteration is used as eigenvalue solver in RSTAB DYNAM Pro

The analytical solution of the equivalent loads corresponding to mode shape 1 of the cantilever beam is compared with results of the multi-modal response spectrum analysis in both RFEM and RSTAB.

Structure File	Program	Analysis Method
0105.01	RFEM 5 - RF-DYNAM Pro	Multi-Modal Response Spectrum Analysis
0105.02	RSTAB 8 - DYNAM Pro	Multi-Modal Response Spectrum Analysis

As can be seen from the following comparisons, excellent agreements of analytical solutions with numerical outputs were achieved in RFEM and RSTAB.

Equivalent Load	Analytical Solution	RFEM	Ratio	RSTAB	Ratio
F_1 [N]	4083.04	4083.02	1.0000	4083.20	1.0000
F_2 [N]	2930.49	2930.52	1.0000	2930.61	1.0000
F_3 [N]	1840.65	1840.70	1.0000	1840.73	1.0000
F_4 [N]	907.23	907.29	0.9999	907.27	1.0000
F_5 [N]	249.55	249.59	0.9998	249.56	1.0000

References

[1] TEDESCO, J., MCDOUGAL, W. and ROSS, C. *Structural Dynamics - Theory and Applications*. Addison-Wesley.

[2] CHOPRA, A. K. *Dynamics of Structures - Theory and Applications to Earthquake Engineering*. Prentice Hall, 2001.

[3] BATHE, K.-J. *Finite Element Procedures*. Prentice Hall, 1996.

[4] DLUBAL SOFTWARE GMBH, *Program Description RF-DYNAM Pro*. 2015.