Program:	RFEM 5
Category:	Geometrically Linear Analysis, Isotropic Nonlinear Elasticity, Isotropic Plasticity, Member, Plate
Verification Example:	0018 – Plastic Bending - Tapered Cantilever

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Description

A tapered cantilever is fully fixed on the left end ($x = 0$) and subjected to a continuous load q according to the **Figure 1**. Small deformations are considered and the self-weight is neglected in this example. Determine the maximum deflection $u_{z,\text{max}}$. The problem is described by the following set of parameters.

Material	Elastic-Plastic	Modulus of Elasticity	E	210000.000	MPa
		Poisson's Ratio	ν	0.000	–
		Shear Modulus	G	105000.000	MPa
		Plastic Strength	f_y	240.000	MPa
Geometry	Cantilever	Length	L	4.000	m
		Width	w	0.005	m
		Left Side Height	h_1	0.250	m
		Right Side Height	h_2	0.150	m
Load		Continuous Load	q	2300.000	Nm^{-1}

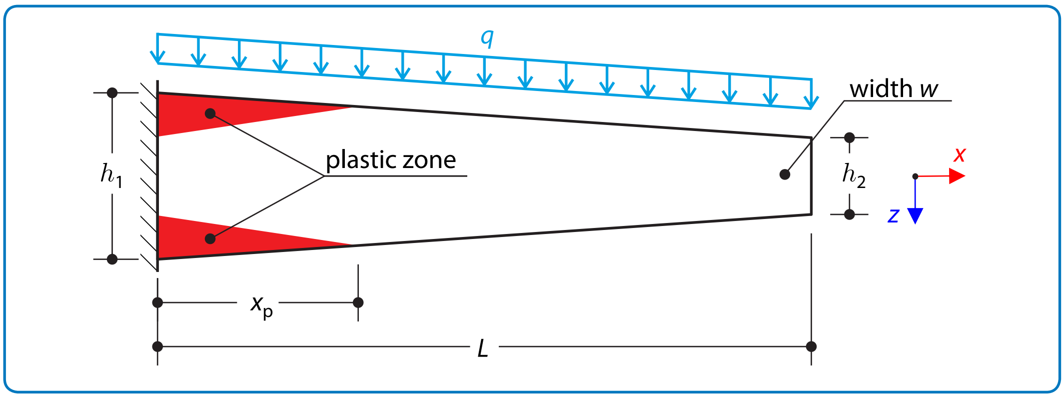

Figure 1: Problem sketch

Analytical Solution

This is more complex variant of the verification example 0006. The tapered cantilever is considered in this case. The function of the cantilever height $h(x)$ is following

$$h(x) = h_1 + \frac{x}{L}(h_2 - h_1) \tag{18 – 1}$$

The bending moment M for the plate under continuous loading q is defined as

$$M = -\frac{q(L-x)^2}{2} \tag{18 – 2}$$

Linear Analysis

Considering linear analysis (only elasticity) the maximum deflection of the structure can be calculated as follows:

$$u_{z,\max} = \int_0^L \frac{q(L-x)^3}{2EI_y(x)} \mathrm{d}x = 71.614 \text{ mm} \tag{18 – 3}$$

Nonlinear Analysis

The quantities of the load are discussed at first. The maximum bending moment obviously occurs on the fully fixed end. The moment M_e when the first yield occurs and the ultimate moment M_p when the structure becomes plastic hinge are calculated as follows

$$M_\mathrm{e} = 2\int_0^{h_1/2} \sigma(z) z w \, \mathrm{d}z = 2\int_0^{h_1/2} \frac{2f_\mathrm{y}}{t} z^2 w \, \mathrm{d}z = \frac{f_\mathrm{y} w h_1^2}{6} = 12.500 \text{ kNm} \tag{18 – 4}$$

$$M_\mathrm{p} = 2\int_0^{h_1/2} \sigma(z) z w \, \mathrm{d}z = 2\int_0^{h_1/2} f_\mathrm{y} z w \, \mathrm{d}z = \frac{f_\mathrm{y} w h_1^2}{4} = 18.750 \text{ kNm} \tag{18 – 5}$$

The corresponding continuous load q_e and q_p then results

$$q_\mathrm{e} = \frac{2M_\mathrm{e}}{L^2} = 2.344 \text{ Nmm}^{-1} \tag{18 – 6}$$

$$q_\mathrm{p} = \frac{2M_\mathrm{p}}{L^2} = 1.563 \text{ Nmm}^{-1} \tag{18 – 7}$$

It is obvious that the continuous load q causes the elastic-plastic state of the plate according to the **Figure 1**. The bending stress is defined according to the following formula

$$\sigma_x(x,z) = -\kappa(x) E z \tag{18 – 8}$$

where $\kappa(x)$ is the curvature defined as $\kappa(x) = \mathrm{d}^2 u_z / \mathrm{d}x^2$ [1]. The elastic-plastic zone length is described by the parameter x_p according to the **Figure 1**. The bending stress quantity on the surface ($z = -h(x)/2$) equals to the plastic strength f_y at the point $x = x_p$, see **Figure 2**. The curvature at this point can be calculated according to the formula

$$\kappa(x_p) = \frac{2f_y}{Eh(x_p)} \qquad (18-9)$$

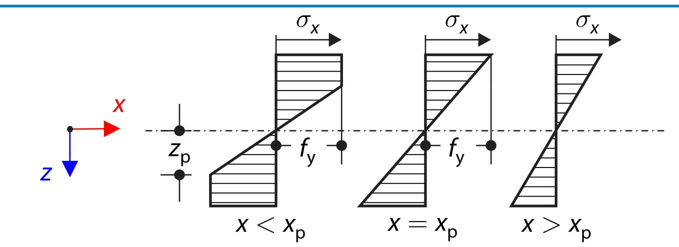

Figure 2: Bending stress distribution

The elastic-plastic moment at the point $x = x_p$ is then

$$M_{ep}(x_p) = \int_{-h(x_p)/2}^{h(x_p)/2} \sigma_x(x_p, z) zw \, dz = 2 \int_0^{h(x_p)/2} -\frac{2f_y}{h(x_p)} z^2 w \, dz = -\frac{f_y w [h(x_p)]^2}{6} \qquad (18-10)$$

The elastic-plastic moment $M_{ep}(x_p)$ (internal force) has to equal to the bending moment $M(x_p)$ (external force).

$$-\frac{f_y w [h(x_p)]^2}{6} = -\frac{q(L - x_p)^2}{2} \qquad (18-11)$$

The elastic-plastic zone length x_p results from this equality as follows

$$x_p = \frac{L - h_1\sqrt{\frac{f_y w}{3q}}}{1 + \frac{h_2 - h_1}{L}\sqrt{\frac{f_y w}{3q}}} = 1048.915 \text{ mm} \qquad (18-12)$$

The curvature κ_e in the elastic zone ($x > x_p$) is described by the Bernoulli-Euler formula

$$\kappa_e = -\frac{M}{EI_y(x)} = \frac{q(L - x)^2}{2EI_y(x)} \qquad (18-13)$$

where $I_y(x)$ is the quadratic moment of the cross-section to the y-axis, which is dependent on the coordinate x thanks to the variable height $h(x)$[1]. The cross-section in the elastic-plastic state is divided into the elastic core and the plastic surface, which is described by the parameter z_p according to the **Figure 2**.

[1] $I_y = \frac{1}{12} w h^3(x)$

$$z_{\mathrm{p}} = \frac{f_{\mathrm{y}}}{\kappa_{\mathrm{p}}(x)E} \tag{18 – 14}$$

The elastic-plastic moment M_{ep} of the cross-section in the elastic-plastic state has to equal to the bending moment M.

$$M_{\mathrm{ep}}(x) = 2\int_{0}^{z_{\mathrm{p}}} -\kappa_{\mathrm{p}}(x)Ez^2w \;\mathrm{d}z + 2\int_{z_{\mathrm{p}}}^{h(x)/2} -f_{\mathrm{y}}zw \;\mathrm{d}z = -\frac{q(L-x)^2}{2} \tag{18 – 15}$$

The curvature κ_{p} in the elastic-plastic zone ($x < x_{\mathrm{p}}$) results from this equality.

$$\kappa_{\mathrm{p}} = \frac{2f_{\mathrm{y}}}{E\sqrt{3}} \frac{1}{\sqrt{h(x)^2 - \frac{2q(L-x)^2}{wf_{\mathrm{y}}}}} \tag{18 – 16}$$

The total deflection of the structure $u_{z,\mathrm{max}}$ is defined as a superposition of the elastic-plastic and the elastic contribution using the Mohr's integral

$$u_{z,\mathrm{max}} = \int_{0}^{x_{\mathrm{p}}} \kappa_{\mathrm{p}}(L-x)\mathrm{d}x + \int_{x_{\mathrm{p}}}^{L} \kappa_{\mathrm{e}}(L-x)\mathrm{d}x = 27.908 + 58.091 = 85.999 \text{ mm} \tag{18 – 17}$$

RFEM 5 Settings

- Modeled in RFEM 5.16.01
- The element size is $l_{\mathrm{FE}} = 0.020$ m for files 0018.01 - 0018.03 and $l_{\mathrm{FE}} = 0.005$ m for files 0018.04 and 0018.05
- Geometrically linear analysis is considered
- The number of increments is 10
- Shear stiffness of the members is neglected

Results

Structure File	Entity	Material model	Description
0018.01	Member	Isotropic Plastic 1D	-
0018.02	Plate	Isotropic Nonlinear Elastic 2D	-
0018.03	Plate	Isotropic Plastic 2D/3D	-
0018.04	Plate	Isotropic Nonlinear Elastic 2D	Variable Thickness
0018.05	Plate	Isotropic Plastic 2D/3D	Variable Thickness
0018.06	Member	Nonlinear Elastic 1D	-

Model	Analytical Solution	RFEM 5	
	$u_{z,max}$ [mm]	$u_{z,max}$ [mm]	Ratio [-]
Isotropic Plastic 1D	85.999	86.215	1.003
Isotropic Nonlinear Elastic 2D, Plate		86.566	1.007
Isotropic Plastic 2D/3D, Plate		84.142	0.978
Isotropic Nonlinear Elastic 2D, Plate, Variable Thickness		83.728	0.974
Isotropic Plastic 2D/3D, Plate, Variable Thickness		83.088	0.966
Isotropic Nonlinear Elastic 1D		86.215	1.003

References

[1] LUBLINER, J. *Plasticity theory*. Berkeley: University of California, 1990.