# 0021 – Plastic Bending with Zero Tensile Strength

## Description

A cantilever is fully fixed on the left end ($x = 0$) and subjected to a transverse force $F$ and an axial force $F_a$ on the right end according to the **Figure 1**. The tensile strength is zero and the behaviour in the compression remains elastic. The problem is described by the following set of parameters.

| | | | | | |
|---|---|---|---|---|---|
| Material | Elastic-Plastic | Modulus of Elasticity | $E$ | 210000.000 | MPa |
| | | Poisson's Ratio | $\nu$ | 0.000 | – |
| | | Shear Modulus | $G$ | 105000.000 | MPa |
| | | Tensile Plastic Strength | $f_t$ | 0.000 | MPa |
| Geometry | Cantilever | Length | $L$ | 2.000 | m |
| | | Width | $w$ | 0.005 | m |
| | | Thickness | $t$ | 0.005 | m |
| Load | | Transverse Force | $F$ | 4.000 | N |
| | | Axial Force | $F_a$ | 5000.000 | N |

Small deformations are considered and the self-weight is neglected in this example. Determine the maximum deflection $u_{z,max}$.

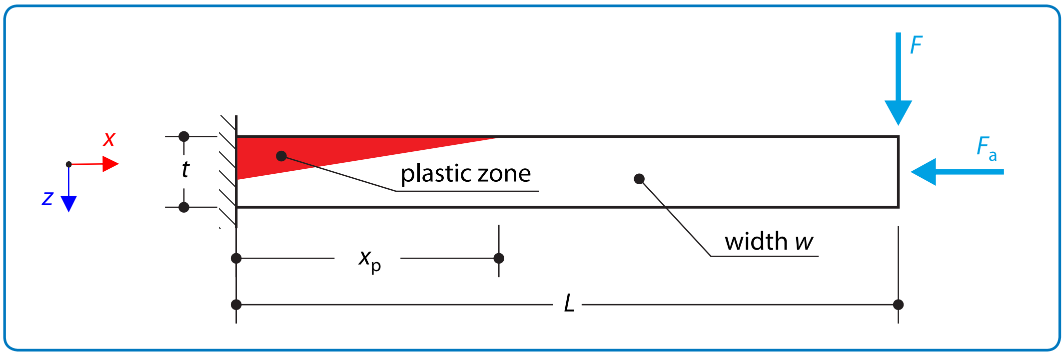


**Figure 1:** Problem sketch

## Analytical Solution

The bending moment $M$ for the cantilever under transverse force $F$ is defined as

$$M = -F(L - x) \qquad (21-1)$$

## Linear Analysis

Considering linear analysis (only elasticity) the maximum deflection of the structure can be calculated as follows:

$$u_{z,\max} = \frac{FL^3}{3EI_y} \approx 0.975 \text{ m} \tag{21 – 2}$$

## Nonlinear Analysis

The transverse force $F$ together with the axial force $F_a$ causes the elastic-plastic of the cantilever according to the **Figure 1**. The elastic-plastic zone length is described by the parameter $x_p$. The stress $\sigma_x$ is composed of the bending stress $\sigma_b$ and the compressive stress $\sigma_c$ according to the **Figure 2** and it is defined according to the following formula

$$\sigma_x(x,z) = -\kappa(x)E(z - z_0(x)) \tag{21 – 3}$$

where $\kappa(x)$ is the curvature defined as $\kappa(x) = \mathrm{d}^2u_z/\mathrm{d}x^2$ [1] and the parameter $z_0(x)$ is defined so that $\sigma_x(x,z_0) = 0$, see **Figure 2**. The first yield occurs, when the bending stress on the top surface on the fixed end reaches the value of the compressive stress.

$$\frac{M(0)}{I_y}\frac{t}{2} = \frac{F_a}{A} \tag{21 – 4}$$

where $I_y$ is the quadratic moment of the cross-section to the $y$-axis[1] and $A$ is the area of the cross-section[2]. The transverse force results $F = 2.08\bar{3}$ N. Thus, the the cantilever under transverse force $F = 4.000$ N is in elastic-plastic state.

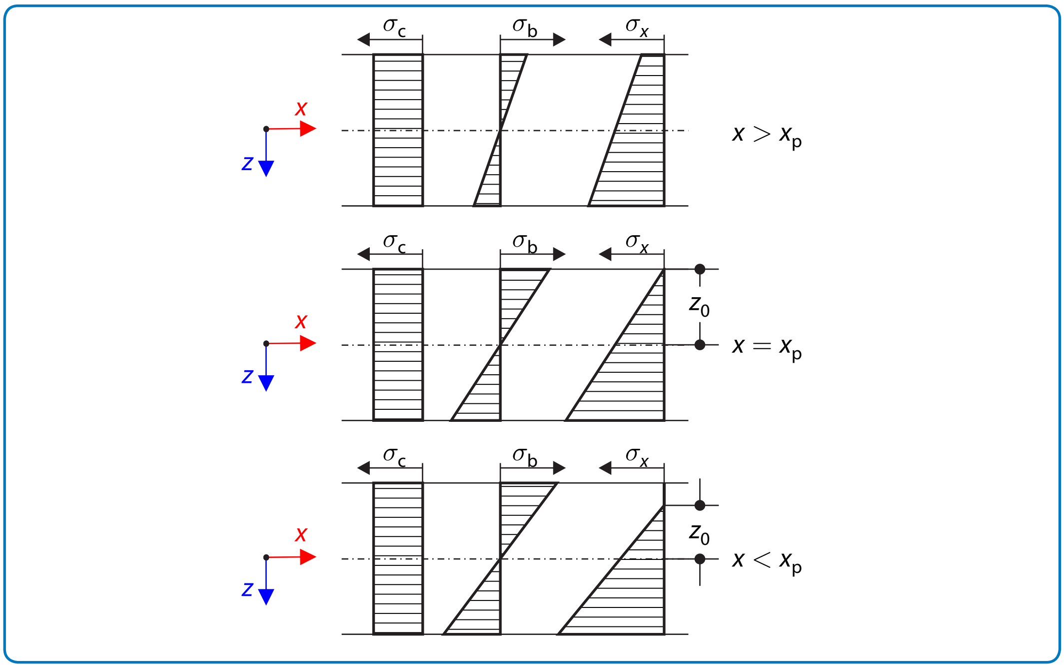


**Figure 2:** Stress distribution

[1] $I_y = \frac{1}{12}wt^3 = 52.08\bar{3}$ mm$^4$
[2] $A = wt = 25.000$ mm$^2$

In the elastic-plastic zone ($x < x_p$) the equilibrium between bending moments and axial forces has to be satisfied.

$$M_{ep} = \int_{z_0}^{t/2} -\kappa_p E(z - z_0)zw \, dz = M \tag{21 – 5}$$

$$N_{ep} = \int_{z_0}^{t/2} -\kappa_p E(z - z_0)w \, dz = -F_a \tag{21 – 6}$$

Solving equations **(21 – 5)** and **(21 – 6)** the curvature in the elastic-plastic zone $\kappa_p$ and the parameter $z_0$ results as follows.

$$\kappa_p(x) = \frac{8F_a^3}{9Ew(F_a t + 2M)^2} \tag{21 – 7}$$

$$z_0(x) = \frac{t}{2} - \frac{3}{2}\frac{F_a t + 2M}{F_a} \tag{21 – 8}$$

The elastic-plastic zone length $x_p$ can be obtained from the equation **(21 – 8)** under the condition $z_0(x_p) = -t/2$.

$$x_p = L - \frac{tF_a}{6F} \approx 958.333 \text{ mm} \tag{21 – 9}$$

The curvature $\kappa_e$ in the elastic zone ($x > x_p$) is described by the Bernoulli-Euler formula

$$\kappa_e = -\frac{M}{EI_y} \tag{21 – 10}$$

The maximum deflection $u_{z,max}$ can be finally calculated according to the following formula

$$u_{z,max} = \int_0^{x_p} \kappa_p(L - x)dx + \int_{x_p}^{L} \kappa_e(L - x)dx \approx 1.232 \text{ m} \tag{21 – 11}$$

## RFEM 5 Settings

- Modeled in RFEM 5.16.01
- The element size is $l_{FE} = 0.020$ m
- Geometrically linear analysis is considered

- The number of increments is 10
- Shear stiffness of the members is neglected

## Results

| Structure File | Entity | Material model | Hypothesis |
|---|---|---|---|
| 0021.01 | Member | Isotropic Nonlinear Elastic 1D | - |
| 0021.02 | Plate | Isotropic Masonry 2D | - |
| 0021.03 | Plate | Nonlinear Elastic 2D/3D | Mohr-Coulomb |
| 0021.04 | Plate | Nonlinear Elastic 2D/3D | Drucker-Prager |
| 0021.05 | Plate | Isotropic Plastic 2D/3D | Mohr-Coulomb |
| 0021.06 | Plate | Isotropic Plastic 2D/3D | Drucker-Prager |

| Model | Analytical Solution | RFEM 5 | |
|---|---|---|---|
| | $u_{z,\text{max}}$ [m] | $u_{z,\text{max}}$ [m] | Ratio [-] |
| Isotropic Nonlinear Elastic 1D | 1.232 | 1.230 | 0.998 |
| Isotropic Masonry 2D | | 1.237 | 1.004 |
| Nonlinear Elastic 2D/3D, Mohr-Coulomb | | 1.237 | 1.004 |
| Nonlinear Elastic 2D/3D, Drucker-Prager | | 1.237 | 1.004 |
| Isotropic Plastic 2D/3D, Mohr-Coulomb | | 1.237 | 1.004 |
| Isotropic Plastic 2D/3D, Drucker-Prager | | 1.236 | 1.003 |

## References

[1] LUBLINER, J. *Plasticity theory*. Berkeley: University of California, 1990.