0096 – Euler Buckling

Description

A strut with circular cross-section is supported according to four basic cases of Euler buckling and it is subjected to pressure force P according to **Figure 1**. Determine the critical load P_{cr}. The problem is described by the following parameters.

Material	Steel	Modulus of Elasticity	E	210000.000	MPa
		Poisson's Ratio	ν	0.300	–
Geometry	Strut	Length	L	2.000	m
		Diameter	d	30.000	mm
Load		Force	P	10.000	kN

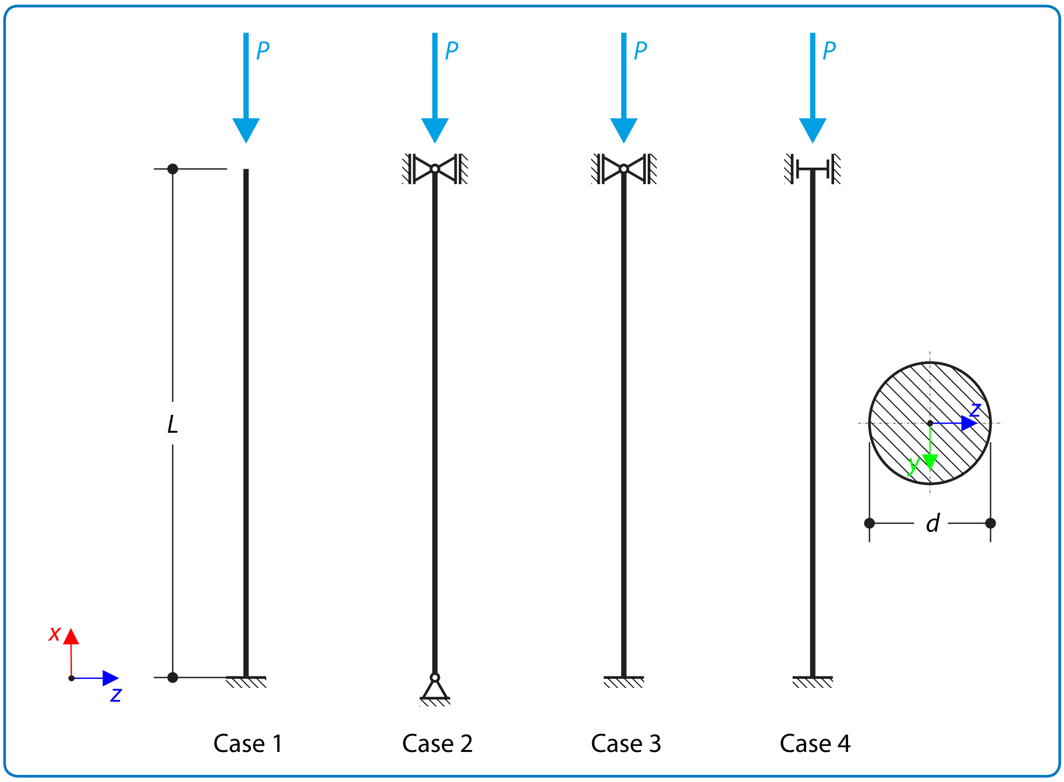

Figure 1: Problem Sketch

Analytical Solution

The governing differential equation for all buckling cases is, see [1],

$$\frac{d^2}{dx^2}y = -\frac{M}{EI}, \tag{96 – 1}$$

where y is the coordinate in buckling direction, M is the bending moment related to the cross-section and I is the moment of inertia[1].

Buckling Case 1

The strut is fixed on one end and free on the other one. Equation **(96 – 1)** can then be rewritten as

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}y = -\frac{P}{EI}(\delta - y), \tag{96 – 2}$$

where δ is the maximum lateral deflection of the tip of the strut. Using the substitution

$$\alpha = \sqrt{\frac{P}{EI}}, \tag{96 – 3}$$

the solution of **(96 – 2)** reads as

$$y = C_1 \cos(\alpha x) + C_2 \sin(\alpha x) + \delta, \tag{96 – 4}$$

Integration constants C_1 and C_2, can be obtained from the following boundary conditions

$$y(0) = 0, \tag{96 – 5}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}y(0) = 0. \tag{96 – 6}$$

The result deflection then reads

$$y = \delta\,(1 - \cos(\alpha x))). \tag{96 – 7}$$

For nonzero δ and for the tip of the strut, the following condition has to be hold

$$\cos(\alpha L) = 0 \quad \Rightarrow \quad \alpha L = k\frac{\pi}{2}, \quad k = 1, 3, 5, \ldots \tag{96 – 8}$$

The minimum value corresponds to $k = 1$ and the critical force P_{cr} is then, by substituting into **(96 – 3)**,

$$P_{cr} = \frac{\pi^2 EI}{4L^2} \approx 5.151 \text{ kN}. \tag{96 – 9}$$

Buckling Case 2

The strut is supported by means of pinned joints, where one is, futhermore, sliding. Equation **(96 – 1)** reads as

[1] Moment of inertia for circular cross-section is $I = \frac{\pi d^4}{64}$.

$$\frac{d^2}{dx^2}y = -\frac{P}{EI}y, \tag{96 – 10}$$

the solution of this differential equation is of the form

$$y = C_1 \cos(\alpha x) + C_2 \sin(\alpha x), \tag{96 – 11}$$

Integration constants C_1 and C_2, can be obtained from the following boundary conditions

$$y(0) = 0, \tag{96 – 12}$$

$$y(L) = 0. \tag{96 – 13}$$

From boundary conditions follows

$$C_2 \sin(\alpha L) = 0. \tag{96 – 14}$$

For nonzero constant C_2 can be written

$$\sin(\alpha L) = 0 \quad \Rightarrow \quad \alpha L = k\pi, \quad k = 1, 2, 3, \ldots \tag{96 – 15}$$

The minimum value corresponds to $k = 1$ and the critical force P_{cr} is then

$$P_{cr} = \frac{\pi^2 EI}{L^2} \approx 20.602 \text{ kN}. \tag{96 – 16}$$

Buckling Case 3

The strut is fixed on one end and supported by means of sliding pinned joint on the other one. Equation **(96 – 1)** can be then rewritten as follows

$$\frac{d^2}{dx^2}y = -\frac{P}{EI}\left(y - \frac{H}{P}(L - x)\right), \tag{96 – 17}$$

where H is the horizontal reaction force in the pinned joint. The solution of this differential equation reads as

$$y = C_1 \cos(\alpha x) + C_2 \sin(\alpha x) + \frac{H}{P}(L - x). \tag{96 – 18}$$

Integration constants C_1 and C_2, can be obtained from the same boundary conditions as **(96 – 5)** and **(96 – 6)**. The result deflection is then

$$y = \frac{H}{P}\left(\frac{1}{\alpha}\sin(\alpha x) - L\cos(\alpha x) + (L - x)\right). \tag{96 – 19}$$

This equation must also satisfy the condition for the second end of the strut **(96 – 13)**, after substitution

$$0 = \frac{H}{P}\left(\frac{1}{\alpha}\sin(\alpha L) - L\cos(\alpha L)\right). \tag{96 – 20}$$

For nonzero horizontal reaction force H, the following equation has to hold

$$\tan(\alpha L) = \alpha L. \tag{96 – 21}$$

This transcendental equation can be solved numerically or grafically with approximate solution

$$P_{cr} \approx 42.147 \text{ kN}. \tag{96 – 22}$$

Buckling Case 4

The strut is fixed on both ends and one end is sliding. Equation **(96 – 1)** takes the form

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}y = -\frac{P}{EI}\left(y - \frac{M_1}{P}\right), \tag{96 – 23}$$

where M_1 is the reaction moment in the fixed support. The solution of this differential equation reads as

$$y = C_1\cos(\alpha x) + C_2\sin(\alpha x) + \frac{M_1}{P}, \tag{96 – 24}$$

Integration constants C_1 and C_2, can be obtained from the boundary conditions **(96 – 5)** and **(96 – 6)**. The general solution then reads

$$y = \frac{M_1}{P}\left(1 - \cos(\alpha x)\right). \tag{96 – 25}$$

This equation must also satisfy the condition for the second end of the strut **(96 – 13)**, after substitution

$$0 = \frac{M_1}{P}\left(1 - \cos(\alpha L)\right). \tag{96 – 26}$$

For nonzero reaction moment M_1, the following equation

$$\cos(\alpha L) = 1 \quad \Rightarrow \quad \alpha L = 2k\pi, \quad k = 1, 2, 3, \ldots \tag{96 – 27}$$

yields a minimum value for $k = 1$ and the critical force P_{cr} is

$$P_{cr} = \frac{4\pi^2 EI}{L^2} \approx 82.409 \text{ kN}. \tag{96 – 28}$$

RFEM 5 and RSTAB 8 Settings

- Modeled in RFEM 5.16.01 and RSTAB 8.16.01
- Element size is $l_{FE} = 0.1$ m
- Isotropic linear elastic material is used
- Lanczos method is used for eigenvalue analysis

Results

Structure File	Program	Buckling Case
0096.01	RFEM 5 - RF-STABILITY	Case 1
0096.02	RFEM 5 - RF-STABILITY	Case 2
0096.03	RFEM 5 - RF-STABILITY	Case 3
0096.04	RFEM 5 - RF-STABILITY	Case 4
0096.05	RSTAB 8 - RSBUCK	Case 1
0096.06	RSTAB 8 - RSBUCK	Case 2
0096.07	RSTAB 8 - RSBUCK	Case 3
0096.08	RSTAB 8 - RSBUCK	Case 4

Buckling Case	Analytical Solution	RFEM 5 – RF-STABILITY		RSTAB 8 – RSBUCK	
	P_{cr} [kN]	P_{cr} [kN]	Ratio [-]	P_{cr} [kN]	Ratio [-]
Case 1	5.151	5.150	1.000	5.153	1.000
Case 2	20.602	20.596	1.000	20.749	1.007
Case 3	42.147	42.107	0.999	43.190	1.025
Case 4	82.409	82.270	0.998	83.324	1.011

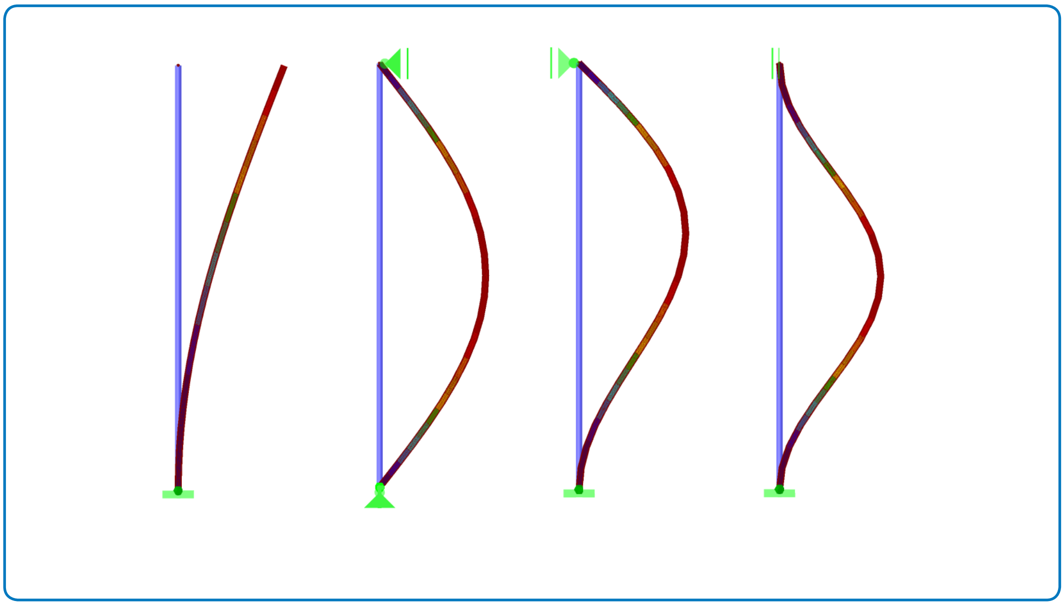

Figure 2: Results in RFEM 5 / RSTAB 8 – First eigenvector.

References

[1] TIMOSHENKO, S. and GERE, J. *Theory of Elastic Stability*. McGraw-Hill Book Company, 1963.