0208 – Helicoid

Description

A membrane is stretched by means of isotropic prestress n in accord with **Figure 1** between two radii in a height h of two concentric cylinders not lying in a plane parallel to the vertical axis. Find the final minimal shape of the membrane – the so-called helicoid, and determine the surface area S of the resulting membrane. The add-on module RF-FORM-FINDING is used for this purpose. Elastic deformations are neglected both in RF-FORM-FINDING and in analytical solution, also self-weight is neglected in this example. The problem is described by the following set of parameters.

Material	Polymer	Modulus of Elasticity	E	692.000	MPa
		Poisson's Ratio	ν	0.442	–
Geometry		Height	h	1.000	m
		Inner Radius	r_1	0.500	m
		Outer Radius	r_2	1.000	m
		Thickness	t	1.000	mm
Load		Prestress	n	1.000	kN/m

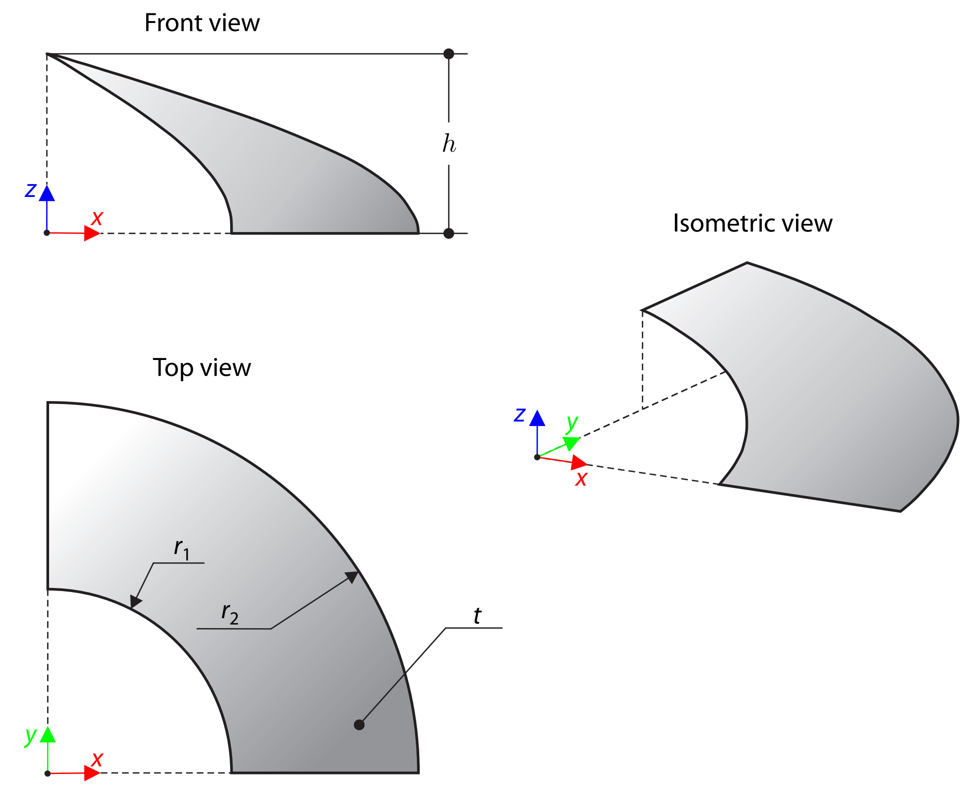

Figure 1: Problem sketch

Analytical Solution

The two radii define four points on two concentric cylinders, while the shortest path between two points not lying above each other on a cylinder is a fraction of a helix[1], and the minimal surface the boundary of which is a helix, is the so-called helicoid.

The (right) helicoid is described by means of following set of parametric equations

$$x = u\cos(v), \tag{208 – 1}$$

$$y = u\sin(v), \tag{208 – 2}$$

$$z = \frac{h}{2\pi} v, \tag{208 – 3}$$

where $v \in [0, 2\pi]$ and $u \in [0, r]$, where r is the radius of bounding cylinder and h the height of a full coil of the helicoid.

Denoting $c = h/2\pi$, the coefficients of the first fundamental form[2] of the helicoid are equal to

$$E = 1, \quad F = 0, \quad G = c^2 + u^2, \tag{208 – 4}$$

yielding the surface area element

$$\mathrm{d}S = \sqrt{EG - F^2}\,\mathrm{d}u\,\mathrm{d}v = \sqrt{c^2 + u^2}\,\mathrm{d}u\,\mathrm{d}v. \tag{208 – 5}$$

Due to symmetry reasons, it suffices to compute one-quarter of the full coil, namely, $v \in [0, \pi/2]$, therefore, for the given parameters of the helicoid part, the surface area is

$$S = \int_0^{\pi/2} \int_{r_1}^{r_2} \sqrt{c^2 + u^2}\,\mathrm{d}u\,\mathrm{d}v \tag{208 – 6}$$

$$= \frac{\pi}{4} \left[u\sqrt{c^2 + u^2} + c^2 \ln\left(\frac{u + \sqrt{c^2 + u^2}}{c} \right) \right]_{r_1}^{r_2} \tag{208 – 7}$$

$$\approx 0.603 \ \mathrm{m}^2.$$

RFEM 5 Settings

- Modeled in RFEM 5.17.01
- The element size is $l_{\mathrm{FE}} = 0.010$ m
- Isotropic linear elastic material model is used

[1] A helix is a curve the tangent of which holds a constant angle with a fixed line, here the vertical axis.

[2] Coefficients E, F and G are obtained from parametrization $X(u, v) = [x(u, v); y(u, v); z(u, v)]^{\mathrm{T}}$ and its derivatives (appropriate tangent vectors) X_u and X_v with respect to u and v. The coefficients E, F and G are defined as $E = X_u \cdot X_u$, $F = X_u \cdot X_v$ and $G = X_v \cdot X_v$.

Results

Structure Files	Program	Modul
0208.01	RFEM 5	RF-FORM-FINDING

The initial surface in RFEM 5 consists of the inner boundary helix part and an arbitrary curve lying on the outer cylinder, here defined as a spline. These two lines are then constrained to deformation in their local *x*–*z* plane.

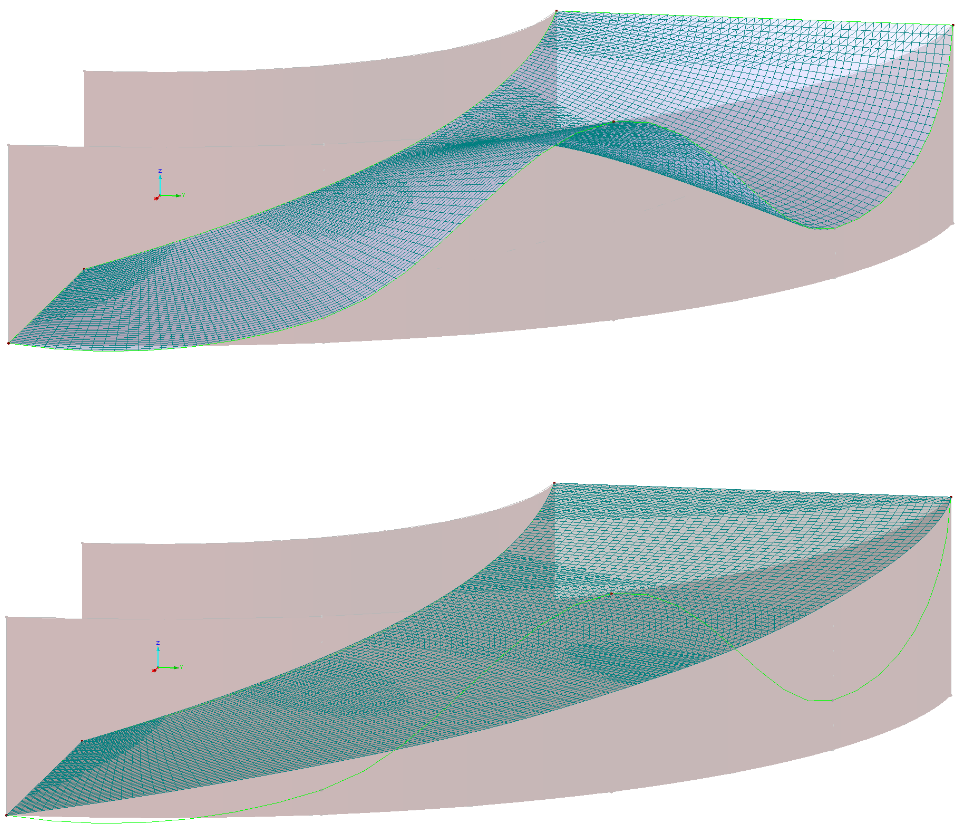

Figure 2: Initial and resulting membrane shape (helicoid) in RFEM 5

Analytical Solution	RFEM 5 – RF-FORM-FINDING	
S [m²]	*S* [m²]	Ratio [-]
0.603	0.604	1.002