**Program:** RFEM 5, RSTAB 8

**Category:** Geometrically Linear Analysis, Isotropic Linear Elasticity, Elastic Foundation, Member

**Verification Example:** 0002 – Cantilever Beam on an Elastic Winkler Foundation

# 0002 – Cantilever Beam on an Elastic Winkler Foundation

## Description

A cantilever beam of length $L$ with rectangular cross-section of height $h$ and width $b$ lying on an elastic Winkler foundation of stiffness $C_{1,z}$ is loaded by a distributed loading $q_z$. Neglecting self-weight, determine the maximum deflection $u_z$ and maximum bending moment $M_y$ of the beam. Calculate the same example also for a plate of the same heigth and width as the cantilever.

| | | | | | |
|---|---|---|---|---|---|
| Material | Isotropic Linear Elastic | Modulus of Elasticity | $E$ | 210.000 | GPa |
| | | Shear Modulus | $G$ | 105.000 | GPa |
| Geometry | Cantilever | Length | $L$ | 4.000 | m |
| | | Height | $h$ | 0.200 | m |
| | | Width | $b$ | 0.005 | m |
| Member Foundation | Winkler Elastic | Stiffness | $C_{1,z}$ | 500.000 | kN/m$^2$ |
| Plate Foundation | | | $C_{u,z} = \frac{C_{1,z}}{b}$ | 100000.000 | kN/m$^3$ |
| Load | Member | Distributed | $q_z$ | 1.000 | kN/m |
| | Plate | Distributed | $q = \frac{q_z}{b}$ | 200.000 | kN/m$^2$ |

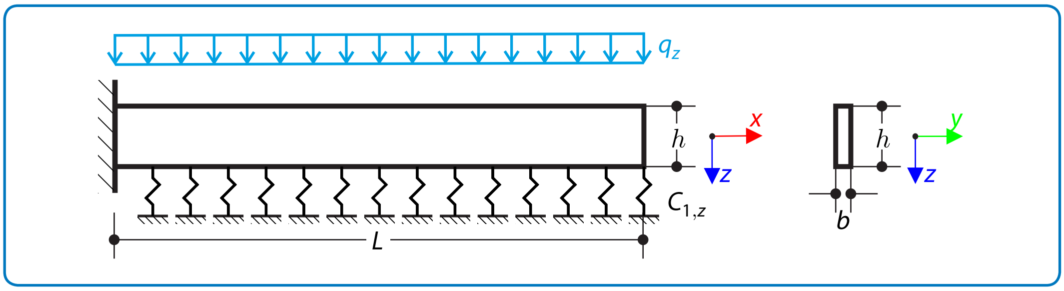


**Figure 1:** Problem sketch

## Analytical Solution

### Member Calculation

The governing differential equation for a beam on an elastic foundation can be expressed as

$$EI_y \frac{d^4 u_z}{dx^4} + C_{1,z} u_z = q_z \qquad (2-1)$$

with the moment of inertia $I_y = \frac{1}{12} b h^3 = 3.3\overline{3} \times 10^{-6}\ \mathrm{m}^4$. Dividing by $EI_y$ and setting $\beta^4 = \frac{C_{1,z}}{4EI_y}$, equation **(2 – 1)** can be rewritten as

$$\frac{d^4 u_z}{dx^4} + 4\beta^4 u_z = \frac{q_z}{EI_y} \tag{2 – 2}$$

The solution of **(2 – 2)** can be obtained as the superposition of the solutions of a particular integral, which can be expressed, assuming $u_z = C = \text{const}$, as

$$0 + 4\beta^4 C = \frac{q_z}{EI_y} = \text{const} \tag{2 – 3}$$

which leads to

$$C = \frac{q_z}{4\beta^4 EI_y} = \frac{q_z}{4\frac{C_{1,z}}{4EI_y} EI_y} = \frac{q_z}{C_{1,z}} \tag{2 – 4}$$

and the solution of the characteristic equation

$$\frac{d^4 u_z}{dx^4} + 4\beta^4 u_z = 0 \tag{2 – 5}$$

To solve the characteristic equation **(2 – 5)**, assume that $u_z = Ae^{\lambda x}$, hence

$$\lambda^4 + 4\beta^4 = 0 \tag{2 – 6}$$

Then the solution for $\lambda$ can be expressed as

$$\lambda^4 = -4\beta^4 \Rightarrow \lambda_{k+1} = \sqrt[4]{(4\beta^4)} \left[ \cos\left(\frac{\pi + 2k\pi}{4}\right) + i \sin\left(\frac{\pi + 2k\pi}{4}\right) \right] \tag{2 – 7}$$

where $k = 0, 1, 2, 3$. Equation **(2 – 7)** can be rewritten for all four variants as

$$\lambda_1(k = 0) = \beta\sqrt{2} \left[ \cos\left(\frac{\pi}{4}\right) + i \sin\left(\frac{\pi}{4}\right) \right] = \beta(1 + i) \tag{2 – 8}$$

$$\lambda_2(k = 1) = \beta\sqrt{2} \left[ \cos\left(\frac{3\pi}{4}\right) + i \sin\left(\frac{3\pi}{4}\right) \right] = \beta(-1 + i) \tag{2 – 9}$$

$$\lambda_3(k = 2) = \beta\sqrt{2} \left[ \cos\left(\frac{5\pi}{4}\right) + i \sin\left(\frac{5\pi}{4}\right) \right] = \beta(-1 - i) \tag{2 – 10}$$

$$\lambda_4(k = 3) = \beta\sqrt{2} \left[ \cos\left(\frac{7\pi}{4}\right) + i \sin\left(\frac{7\pi}{4}\right) \right] = \beta(1 - i) \tag{2 – 11}$$

Therefore, the solution $u_z$ of **(2 – 5)** takes the form

$$u_z = \sum_{i=1}^{4} A_i e^{\lambda_i x} \tag{2 – 12}$$

Substituting equations **(2 – 8)**–**(2 – 11)**, **(2 – 12)** can be rewritten as

$$u_z = A_1 e^{\beta x(1+i)} + A_2 e^{\beta x(-1+i)} + A_3 e^{\beta x(-1-i)} + A_4 e^{\beta x(1-i)} = \qquad (2-13)$$

$$e^{\beta x}\left(A_1 e^{\beta x i} + A_4 e^{-\beta x i}\right) + e^{-\beta x}\left(A_2 e^{\beta x i} + A_3 e^{-\beta x i}\right) \qquad (2-14)$$

Incorporating $e^{\beta x i} = \cos(\beta x) + i\sin(\beta x)$ into **(2 – 13)** yields

$$u_z = e^{\beta x}(C_1\cos(\beta x) + C_2\sin(\beta x)) + e^{-\beta x}(C_3\cos(\beta x) + C_4\sin(\beta x)) = \qquad (2-15)$$

$$\cos(\beta x)\left(C_1 e^{\beta x} + C_3 e^{-\beta x}\right) + \sin(\beta x)\left(C_2 e^{\beta x} + C_4 e^{-\beta x}\right) \qquad (2-16)$$

which can be further simplified using a new set of unknowns and the definition of hyperbolic functions

$$u_z = \cos(\beta x)(D_1\cosh(\beta x) + D_2\sinh(\beta x)) + \sin(\beta x)(D_3\cosh(\beta x) + D_4\sinh(\beta x)) \qquad (2-17)$$

The final solution of equation **(2 – 2)** is constructed by the superposition of the solutions **(2 – 4)** and **(2 – 17)**

$$u_z = \cos(\beta x)(D_1\cosh(\beta x) + D_2\sinh(\beta x)) + \sin(\beta x)(D_3\cosh(\beta x) + D_4\sinh(\beta x)) + \frac{q_z}{C_{1,z}} \qquad (2-18)$$

To obtain values for constants $D_1$–$D_4$, four cantilever boundary conditions have to be applied

$$1)\quad u_z(0) = 0 \qquad (2-19)$$

$$2)\quad \frac{\mathrm{d}u_z}{\mathrm{d}x}(0) = 0 \qquad (2-20)$$

$$3)\quad M_y(L) = EI_y\frac{\mathrm{d}^2u_z}{\mathrm{d}x^2}(L) = 0 \Rightarrow \frac{\mathrm{d}^2u_z}{\mathrm{d}x^2}(L) = 0 \qquad (2-21)$$

$$4)\quad V_z(L) = EI_y\frac{\mathrm{d}^3u_z}{\mathrm{d}x^3}(L) = 0 \Rightarrow \frac{\mathrm{d}^3u_z}{\mathrm{d}x^3}(L) = 0 \qquad (2-22)$$

which leads to

$$1)\quad u_z(0) = D_1 + \frac{q_z}{C_{1,z}} = 0 \Rightarrow D_1 = -\frac{q_z}{C_{1,z}} \qquad (2-23)$$

$$2)\quad \frac{\mathrm{d}u_z}{\mathrm{d}x}(0) = \beta(D_2 + D_3) = 0 \Rightarrow D_2 = -D_3 \qquad (2-24)$$

$$3)\quad \frac{\mathrm{d}^2u_z}{\mathrm{d}x^2}(L) = -2\beta^2(D_1 ss_h + D_2 sc_h - D_3 cs_h - D_4 cc_h) = 0 \qquad (2-25)$$

$$4)\quad \frac{d^3u_z}{dx^3}(L) = -2\beta^3(D_1 sc_h + D_2 ss_h + D_1 cs_h + D_2 cc_h - D_3 cc_h - D_4 cs_h + D_3 ss_h + D_4 sc_h) = 0 \qquad (2-26)$$

where $s = \sin(\beta L)$, $c = \cos(\beta L)$, $s_h = \sinh(\beta L)$, and $c_h = \cosh(\beta L)$. Substituting **(2 – 23)** and **(2 – 24)** into **(2 – 25)** and **(2 – 26)**, the following relations are obtained

$$3)\quad -\frac{q_z}{C_{1,z}}(ss_h) - D_3(sc_h + cs_h) - D_4(cc_h) = 0 \qquad (2-27)$$

$$4)\quad -\frac{q_z}{C_{1,z}}(sc_h + cs_h) - D_3(ss_h + cc_h + cc_h - ss_h) - D_4(cs_h - sc_h) = 0 \qquad (2-28)$$

Combining **(2 – 27)**, **(2 – 28)** yields

$$\begin{bmatrix} sc_h + cs_h & cc_h \\ 2cc_h & cs_h - sc_h \end{bmatrix} \begin{bmatrix} D_3 \\ D_4 \end{bmatrix} = \begin{bmatrix} -\frac{q_z}{C_{1,z}} ss_h \\ -\frac{q_z}{C_{1,z}}(sc_h + cs_h) \end{bmatrix} \qquad (2-29)$$

Solving **(2 – 29)** leads to the coefficients $D_3$ and $D_4$ in the form

$$D_3 = -\frac{q_z}{C_{1,z}}\left(\frac{cs + s_h c_h}{2 + c_h^2}\right) \qquad (2-30)$$

$$D_4 = -\frac{q_z}{C_{1,z}}\left(\frac{c^2 - c_h^2}{c^2 + c_h^2}\right) \qquad (2-31)$$

Finally, substituting equations **(2 – 23)**, **(2 – 24)**, **(2 – 30)**, and **(2 – 31)** into **(2 – 18)** and setting $x = L$, the value for the maximum deflection $u_z$ is obtained

$$u_{z,\max} = u_z(L) = c(D_1 c_h + D_2 s_h) + s(D_3 c_h + D_4 s_h) + \frac{q_z}{C_{1,z}} = 2.498 \text{ mm} \qquad (2-32)$$

Similarly, setting $x = 0$ and substituting **(2 – 25)** and **(2 – 31)** into **(2 – 21)** gives the value for the maximum bending moment $M_y$

$$M_{y,\max} = M_y(0) = -EI_y \frac{d^2u_z}{dx^2}(0) = -2EI_y\beta^2 D_4 = -1.146 \text{ kNm} \qquad (2-33)$$

## Plate Calculation

The cantilever is also calculated using plate elements of width $b$ and height $h$ on a Pasternak foundation. The example yields the same numerical results, so the theory is identical. The parameter $C_{2,z}$ describing the Pasternak foundation for plates that yields the same results is equal to $C_{u,z} = \frac{C_{1,z}}{b} = 100000 \text{ kN/m}^3$.

Note that, in order to approximate the member solution exactly, the Poisson ratio is zero.

## RFEM 5 and RSTAB 8 Settings

- Modeled in version RFEM 5.16.01 and RSTAB 8.16.01
- The element sizes are $l_{FE} = 0.400$ m (member) and $l_{FE} = 0.100$ m (plate)
- Geometrically linear analysis is considered
- Isotropic linear elastic material model is used
- The Kirchhoff plate theory is used
- Shear stiffness of members is deactivated

## Results

| Structure File | Entity | Program |
|---|---|---|
| 0002.01 | Member | RFEM 5 |
| 0002.02 | Member | RSTAB 8 |
| 0002.03 | Plate | RFEM 5 |

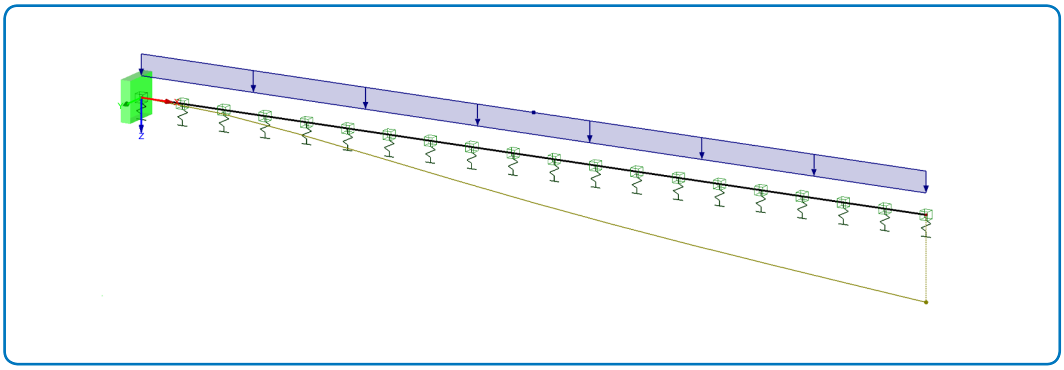

**Figure 2:** RFEM 5 Model – Member

As seen from the following comparisons, excellent agreement between the analytical solutions and numerical outputs has been achieved.

| Analytical Solution | RFEM 5 (Member) | | RSTAB 8 (Member) | | RFEM 5 (Plate) | |
|---|---|---|---|---|---|---|
| $u_{z,max}$ [mm] | $u_{z,max}$ [mm] | Ratio [-] | $u_{z,max}$ [mm] | Ratio [-] | $u_{z,max}$ [mm] | Ratio [-] |
| 2.498 | 2.498 | 1.000 | 2.498 | 1.000 | 2.495 | 0.999 |

| Analytical Solution | RFEM 5 (Member) | | RSTAB 8 (Member) | | RFEM 5 (Plate) | |
|---|---|---|---|---|---|---|
| $M_{y,max}$ [kNm] | $M_{y,max}$ [kNm] | Ratio [-] | $M_{y,max}$ [kNm] | Ratio [-] | $m_{x,max} \times b$ [kNm] | Ratio [-] |
| $-1.146$ | $-1.146$ | 1.000 | $-1.146$ | 1.000 | $-1.139$ | 0.994 |