0104 – Cantilever Beam (SDOF) with Periodic Excitation

Description

A cantilever I-beam of length L, height h, and width b with a nodal mass m on its free end is considered, the self-weight is neglected. This single-degree-of-freedom system (SDOF) is excited by a periodic oscillation $F(t) = F_0 \sin(\Omega t)$ with an angular frequency Ω at its free end. The deflection $u_z(t)$ of the beam is determined.

Material	Isotropic Linear Elastic	Modulus of Elasticity	E	210.000	GPa
		Shear Modulus	G	81.000	GPa
Structure	Cantilever Beam	Length	L	1.000	m
		Load	F_0	1.000	kN
	Cross Section IPE 80	Depth	d	0.080	m
		Width	b	0.046	m
		Web Thickness	t_w	0.004	m
		Flange Thickness	t_f	0.005	m
		Radius	r	0.005	mm
		Moment of Inertia	I_y	8.014×10^{-7}	m^4
SDOF System	Nodal Mass		m	100.000	kg
	Damping	Lehr's Damping	D	0.010	–
Periodic Excitation	Frequency		Ω	10.000	rad/s
	Amplitude		F_0	1.000	kN
	Initial Conditions	Displacement	u_0	0.000	m
		Velocity	v_0	0.000	m/s

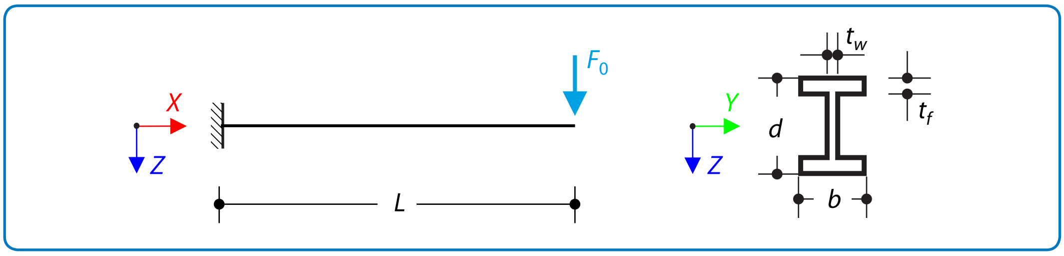

Figure 1: Problem sketch

Analytical Solution

The following terms and relations are based on the work of J. W. Tedesco [1].

The stiffness k of the cantilever beam is equal to

$$k = \frac{3EI_y}{L^3} \approx 504.882 \text{ kN/m} \tag{104 – 1}$$

from where the circular frequency of the undamped SDOF system is then calculated as

$$\omega = \sqrt{\frac{k}{m}} \approx 71.055 \text{ rad/s} \tag{104 – 2}$$

which corresponds to the natural frequency

$$f = \frac{\omega}{2\pi} \approx 11.309 \text{ Hz} \tag{104 – 3}$$

SDOF System without Damping

The motion of a freely vibrating SDOF system is described by the homogeneous second-order ordinary differential equation

$$m\,\ddot{u}_h + k\,u_h = 0 \tag{104 – 4}$$

the general solution of which is

$$u_h(t) = A\,\sin(\omega t) + B\,\cos(\omega t) \tag{104 – 5}$$

where the constants A and B are determined from the initial conditions.

The equation of motion for an SDOF system under forced vibration, excited by a harmonic function with frequency Ω, is given by

$$m\,\ddot{u}_p + k\,u_p = F_0 \sin(\Omega t) \tag{104 – 6}$$

The resulting particular solution for this differential equation is

$$u_p(t) = \frac{F_0/k}{1-(\Omega/\omega)^2}\sin(\Omega t) = \frac{u_{st}}{1-\eta^2}\sin(\Omega t) \tag{104 – 7}$$

where $u_{st} = F_0/k$ is the equivalent static deflection that would result from a force F_0,

$$u_{st} = \frac{F_0}{k} \approx 1.981 \text{ mm} \tag{104 – 8}$$

and $\eta = \Omega/\omega$ is the so-called frequency ratio,

$$\eta = \frac{\Omega}{\omega} \approx 0.141 \tag{104 – 9}$$

Recall that the dynamic response factor R_d, which is the ratio between the static and dynamic displacement, is for the undamped SDOF system defined as

$$R_d = \frac{1}{1-\eta^2} \tag{104 – 10}$$

The complete solution of this SDOF system is the sum of the homogeneous and particular solution, in particular, the solution for displacement $u(t)$, velocity $\dot{u}(t)$ and acceleration $\ddot{u}(t)$ is given by

$$u(t) = A\,\sin(\omega t) + B\,\cos(\omega t) + \frac{u_{st}}{1-\eta^2}\sin(\Omega t) \tag{104 – 11}$$

$$\dot{u}(t) = A\,\omega\cos(\omega t) - B\,\omega\sin(\omega t) + \frac{u_{st}}{1-\eta^2}\Omega\cos(\Omega t) \tag{104 – 12}$$

$$\ddot{u}(t) = -\left(A\,\omega^2\,\sin(\omega t) + B\,\omega^2\,\cos(\omega t) + \frac{u_{st}}{1-\eta^2}\Omega^2\sin(\Omega t)\right) \tag{104 – 13}$$

The constants A and B from the homogeneous part of the solution are determined from the initial conditions $u(0) = u_0$ and $\dot{u}(0) = v_0$, namely

$$A = -\frac{\eta\,u_{st}}{1-\eta^2} \tag{104 – 14}$$

$$B = 0 \tag{104 – 15}$$

Inserting A and B into **(104 – 11)**—**(104 – 13)**, the final solution for this SDOF system reads as

$$u(t) = \frac{u_{st}}{1-\eta^2}\left[\sin(\Omega t) - \eta\,\sin(\omega t)\right] \tag{104 – 16}$$

$$\dot{u}(t) = \frac{\Omega\,u_{st}}{1-\eta^2}\left[\cos(\Omega t) - \cos(\omega t)\right] \tag{104 – 17}$$

$$\ddot{u}(t) = \frac{\Omega\,u_{st}}{1-\eta^2}\left[\omega\,\sin(\omega t) - \Omega\,\sin(\Omega t)\right] \tag{104 – 18}$$

SDOF System with Viscous Damping

The equation of motion for a freely vibrating damped SDOF system is given by

$$m\,\ddot{u}_h + c\,\dot{u}_h + k\,u_h = 0 \tag{104 – 19}$$

where c is the viscous-damping coefficient. The relation between Lehr's damping D and the viscous damping coefficient c reads as

$$D = \frac{c}{C_c} = \frac{c}{2\,m\,\omega} \tag{104 – 20}$$

where $C_c = 2m\,\omega$ is the so-called critical-damping constant. The presented SDOF system exhibits subcritical damping, as $D < 1$. Then, the solution to **(104 – 19)** is given by

$$u_h(t) = e^{-D\omega t}\left[A\,\sin(\omega_d t) + B\,\cos(\omega_d t)\right] \tag{104 – 21}$$

where ω_d is the damped circular frequency,

$$\omega_d = \sqrt{1 - D^2}\,\omega \approx 71.051\text{ rad} \tag{104 – 22}$$

The specific values of the constants A and B follow again from the initial conditions.

The equation of motion damped SDOF system under forced vibration, excited by a harmonic function with frequency Ω, is given by

$$m\,\ddot{u}_p + c\,\dot{u}_p + k\,u_p = F_0\,\sin(\Omega t) \tag{104 – 23}$$

A particular solution of this differential equation is

$$u_p(t) = R_d\,u_{st}\,\sin(\Omega t - \Psi) \tag{104 – 24}$$

where u_{st} is the equivalent static deflection **(104 – 8)**, and η the frequency ratio **(104 – 9)**. The dynamic response factor R_d is given by

$$R_d = \frac{1}{\sqrt{(1-\eta^2)^2 + (2D\eta)^2}} \tag{104 – 25}$$

which represents the ratio between static and dynamic amplitude, for further details see [2]. Note that for $D = 0$ the equation is identical to **(104 – 10)**. The phase angle Ψ represents the lag of the response behind the periodic excitation,

$$\Psi = \tan^{-1}\left(\frac{2D\eta}{1-\eta^2}\right) \approx 2.871 \times 10^{-3}\text{ rad} \tag{104 – 26}$$

The complete solution of this SDOF system is the sum of the homogeneous and particular solution, namely

$$u(t) = e^{-D\omega t}\left[A\,\sin(\omega_d t) + B\,\cos(\omega_d t)\right] + R_d\,u_{st}\,\sin(\Omega t - \Psi) \tag{104 – 27}$$

$$\dot{u}(t) = e^{-D\omega t}\left[-\left[A\,D\,\omega + B\,\omega_d\right]\sin(\omega_d\,t) + \left[A\,\omega_d - B\,D\,\omega\right]\cos(\omega_d\,t)\right] + R_d\,u_{st}\Omega\cos(\Omega t - \Psi) \tag{104 – 28}$$

$$\ddot{u}(t) = e^{-D\omega t}\left[\left[-A(D^2\,\omega^2 + \omega_d^2) + 2\,B\,D\,\omega\,\omega_d\right]\sin(\omega_d\,t) - \left[2\,A\,D\,\omega\,\omega_d - B(D^2\,\omega^2 + \omega_d^2)\right]\cos(\omega_d\,t)\right] - R_d\,u_{st}\Omega^2\sin(\Omega t - \Psi) \tag{104 – 29}$$

where

$$A = -R_d\,\frac{u_{st}}{\omega_d}\left[D\,\omega\,\sin(-\Psi) + \Omega\,\cos(-\Psi)\right] \tag{104 – 30}$$

$$B = -R_d\,u_{st}\,\sin(-\Psi) \tag{104 – 31}$$

are—again—determined from the initial conditions $u(0) = u_0$ and $\dot{u}(0) = v_0$.

RFEM 5 and RSTAB 8 Settings

- Modeled in version RFEM 5.05.0019 and RSTAB 8.04.0019
- The member is not divided into finite elements (RFEM) nor internal nodes (RSTAB)
- Linear dynamic analysis is considered, modal analysis and direct integration (Newmark method) are used
- The time increment is $\Delta t = 1 \times 10^{-3}$ s for the implicit Newmark method
- Isotropic linear elastic material model is used
- Shear stiffness of members is deactivated

Analytical solutions are compared with the results of direct integration and modal analysis in both RFEM and RSTAB. The displacements and the accelerations at time steps where the maximum displacements at the free end of the cantilever beam occur are compared. The time step $\Delta t = 1 \times 10^{-3}$ s for the implicit Newmark method has been chosen with recommendations given in [3],

$$\Delta t = \frac{1}{20\,f} = 4.420 \times 10^{-3}\ \text{s} \tag{104 – 32}$$

Results

In RSTAB DYNAM Pro and RFEM RF-DYNAM Pro, both direct integration and modal analysis are available. The values of the displacement u_z and the acceleration $\ddot{u}_z$ are compared with the analytical solution, separate for the undamped and the SDOF system with viscous damping, in the tables below.

Results of the SDOF System without Damping

Structure File	Program	Analysis Method	Dynamic Load Case
0104.01	RFEM 5 – RF-DYNAM Pro	Modal Analysis	DLC1
0104.01	RFEM 5 – RF-DYNAM Pro	Direct Integration	DLC2
0104.02	RSTAB 8 – DYNAM Pro	Modal Analysis	DLC1
0104.02	RSTAB 8 – DYNAM Pro	Direct Integration	DLC2

As seen from the following comparisons, excellent agreements of the analytical and numerical solutions were achieved.

Time	Analytical Solution		RFEM 5 - Modal Analysis $\Delta t = 1 \times 10^{-3}$ s			
t [s]	u_z [mm]	$\ddot{u}_z$ [m/s^2]	u_z [mm]	Ratio [–]	$\ddot{u}_z$ [m/s^2]	Ratio [–]
0.155	2.305	−1.638	2.305	1.000	−1.638	1.000
0.775	2.293	−1.631	2.294	1.000	−1.634	0.998
1.395	2.266	−1.616	2.268	0.999	−1.625	0.994
2.015	2.224	−1.592	2.228	0.998	−1.610	0.989

Time	Analytical Solution		RFEM 5 - Direct Integration $\Delta t = 1 \times 10^{-3}$ s			
t [s]	u_z [mm]	$\ddot{u}_z$ [m/s^2]	u_z [mm]	Ratio [–]	$\ddot{u}_z$ [m/s^2]	Ratio [–]
0.155	2.305	−1.638	2.305	1.000	−1.638	1.000
0.775	2.293	−1.631	2.294	1.000	−1.634	0.998
1.395	2.266	−1.616	2.268	0.999	−1.625	0.994
2.015	2.224	−1.592	2.228	0.998	−1.610	0.989

In RSTAB, the modal analysis provides the exact analytical solution, and also the direct integration provides very good results.

Time	Analytical Solution		RSTAB 8 - Modal Analysis $\Delta t = 1 \times 10^{-3}$ s			
t [s]	u_Z [mm]	$\ddot{u}_z$ [m/s^2]	u_z [mm]	Ratio [–]	$\ddot{u}_Z$ [m/s^2]	Ratio [–]
0.155	2.305	−1.638	2.305	1.000	−1.638	1.000
0.775	2.293	−1.631	2.293	1.000	−1.631	1.000
1.395	2.266	−1.616	2.266	1.000	−1.616	1.000
2.015	2.224	−1.592	2.224	1.000	−1.592	1.000

Time	Analytical Solution		RSTAB 8 - Direct Integration $\Delta t = 1 \times 10^{-3}\ s$			
t [s]	u_Z [mm]	$\ddot{u}_Z$ [m/s^2]	u_z [mm]	Ratio [–]	$\ddot{u}_Z$ [m/s^2]	Ratio [–]
0.155	2.305	−1.638	2.305	1.000	−1.638	1.000
0.775	2.293	−1.631	2.294	1.000	−1.634	0.998
1.395	2.266	−1.616	2.268	0.999	−1.624	0.995
2.015	2.224	−1.592	2.227	0.999	−1.609	0.989

All results, achieved in RFEM and RSTAB are compared graphically with the analytical solution, the difference can be hardly seen.

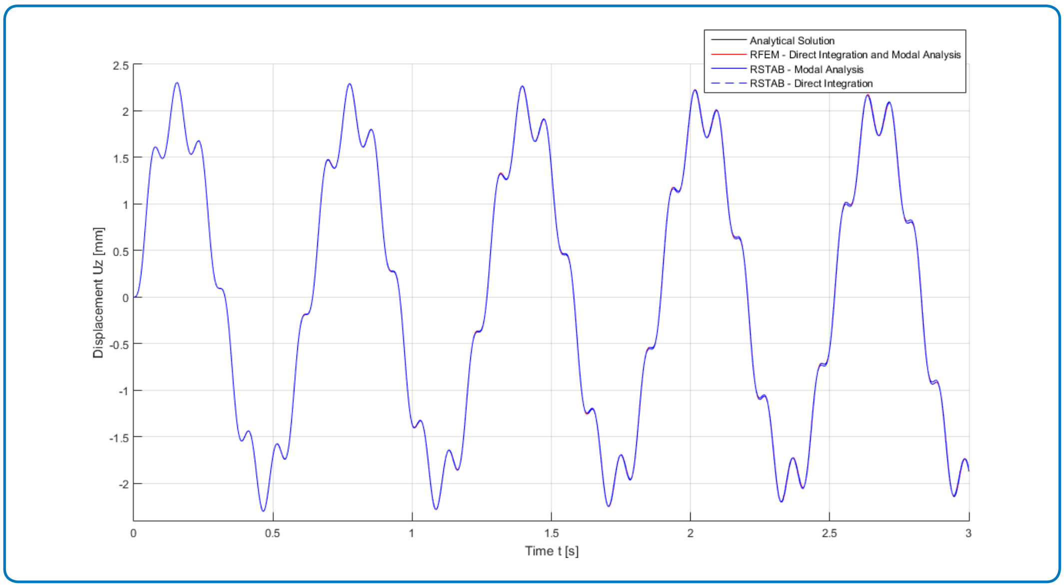

Figure 2: Displacement u_Z versus time t, the analytical solution compared with RFEM and RSTAB. The differences can be hardly seen, the curves are on top of each other.

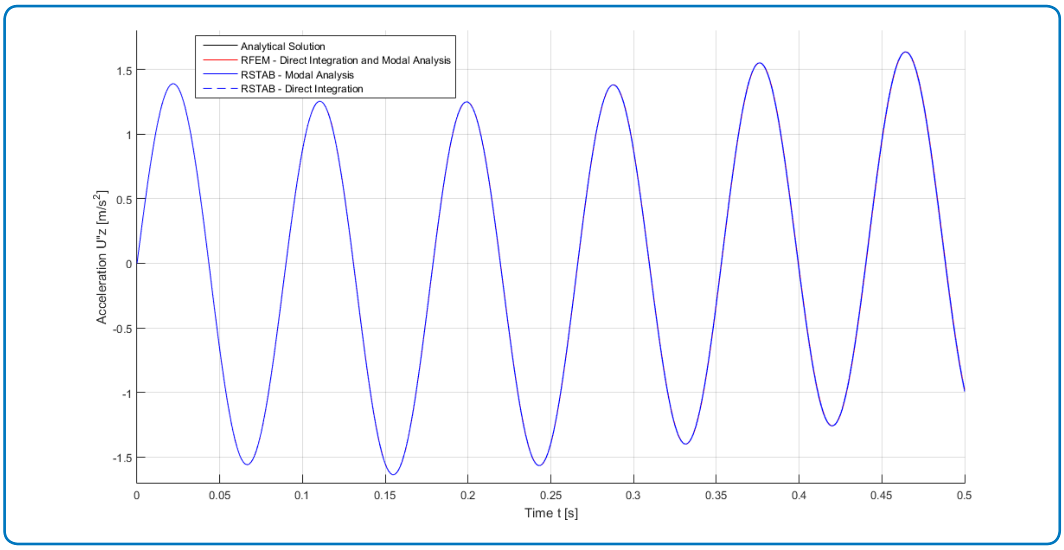

Figure 3: Acceleration $\ddot{u}_Z$ versus time t, the analytical solution compared with RFEM and RSTAB. The differences can be hardly seen, the curves are on top of each other.

Results of the Damped SDOF System

Structure File	Program	Analysis Method	Dynamic Load Case
0104.03	RFEM 5	Modal Analysis	DLC1
0104.04	RSTAB 8	Modal Analysis	DLC1

As can be seen from the following comparisons, good agreements of analytical solutions with numerical outputs were achieved in RFEM. In case of a damped system, a smaller time step would increase the accuracy further.

Time	Analytical Solution		RFEM 5 - Modal Analysis $\Delta t = 1 \times 10^{-3}$ s			
t [s]	u_Z [mm]	$\ddot{u}_Z$ [m/s^2]	u_z [mm]	Ratio [–]	$\ddot{u}_Z$ [m/s^2]	Ratio [–]
0.155	2.275	−1.487	2.275	1.000	−1.488	0.999
0.776	2.174	−1.013	2.174	1.000	−1.022	0.992
1.399	2.094	−0.671	2.096	0.999	−0.691	0.971
2.024	2.032	−0.409	2.035	0.999	−0.437	0.936

In RSTAB, the modal analysis provides the exact analytical solution, and also the direct integration provides good results. To increase the accuracy even further a smaller time step would be required.

Time	Analytical Solution		RSTAB 8 - Modal Analysis $\Delta t = 1 \times 10^{-3}$ s			
t [s]	u_Z [mm]	$\ddot{u}_Z$ [m/s^2]	u_z [mm]	Ratio [–]	$\ddot{u}_Z$ [m/s^2]	Ratio [–]
0.155	2.275	−1.487	2.275	1.000	−1.488	0.999
0.776	2.174	−1.013	2.174	1.000	−1.018	0.995
1.399	2.094	−0.671	2.094	1.000	−0.681	0.985
2.024	2.032	−0.409	2.032	1.000	−0.420	0.974

All results, achieved in RFEM and RSTAB are compared graphically with the analytical solution, the difference can be hardly seen.

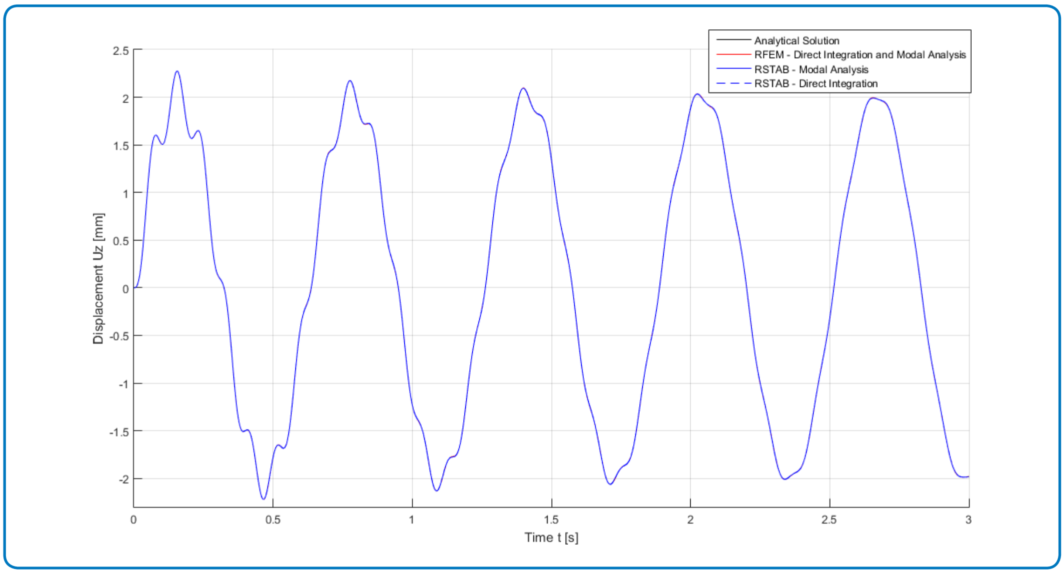

Figure 4: Displacement u_Z versus time t, the analytical solution compared with RFEM and RSTAB. The differences can be hardly seen, the curves are on top of each other.

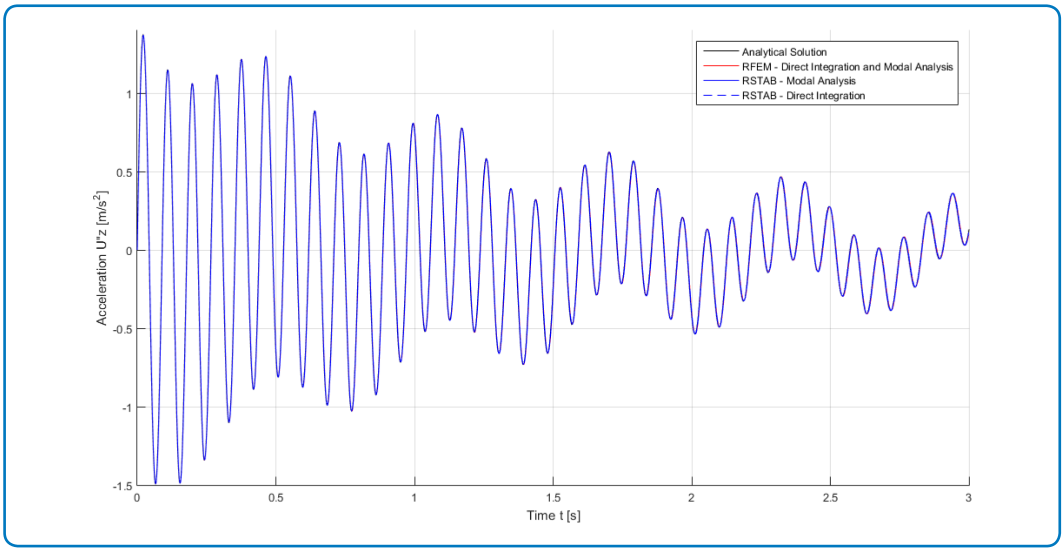

Figure 5: Acceleration $\ddot{u}_Z$ versus time t, the analytical solution compared with RFEM and RSTAB. The differences can be hardly seen, the curves are on top of each other.

References

[1] TEDESCO, J., MCDOUGAL, W. and ROSS, C. *Structural Dynamics - Theory and Applications*. Addison-Wesley, 1999.

[2] CHOPRA, A. K. *Dynamics of Structures - Theory and Applications to Earthquake Engineering*. Prentice Hall, 2001.

[3] STELZMANN, U., GROTH, C. and MÜLLER, G. *FEM für Praktiker - Band 2: Strukturdynamik*. Expert Verlag, 2008.