0300 – Drag Force on a Sphere

Description

A sphere is subjected to uniform flow of viscous fluid according to **Figure 1**. The velocity u of the fluid is considered at infinity. The goal is to determine the drag force F_x. The parameters of the problem are set so that the Reynolds number is small and the radius of the sphere is also small, thus the theoretical solution can be reached – Stokes flow (G. G. Stokes 1851). The problem is described by the following set of parameters.

Fluid Properties	Dynamic Viscosity	μ	0.00012	Pas
	Density	ρ	1.200	kg/m^3
Geometry	Sphere Radius	R	0.0005	m
Load	Fluid Velocity	u	0.001	m/s

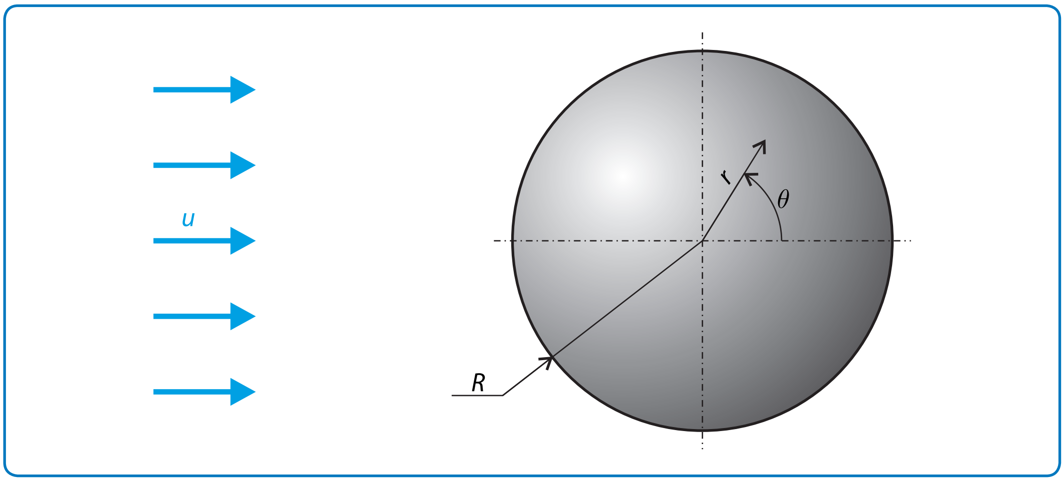

Figure 1: Problem sketch

Analytical Solution

Analytical solution is based on the theory introduced in [1]. Motion of incompressible fluids is described by the Navier-Stokes equation

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} = -\frac{1}{\rho}\nabla p + \frac{\eta}{\rho} \triangle \mathbf{v}, \qquad (300-1)$$

where $\mathbf{v}$ is the flow velocity vector field, p is the pressure and ρ is the density. The incompressibility condition follows

$$\operatorname{div} \mathbf{v} = 0. \qquad (300-2)$$

Finally, to completely determine the motion of the fluid, zero-velocity boundary condition is considered at the surface of the sphere, namely, in polar coordnates,

$$\mathbf{v}(R) = 0. \tag{300 – 3}$$

The Stokes flow assumes small values of Reynolds number ($Re \ll 1$). In this case, it is

$$Re = \frac{vL}{\nu} = \frac{\rho u R}{\eta} = 0.005, \tag{300 – 4}$$

where L is the characteristic dimension and ν is the kinematic viscosity

$$\nu = \frac{\eta}{\rho} = 0.0001\ \mathrm{m^2/s}. \tag{300 – 5}$$

Velocity Vector Field

Assuming an incompressible fluid, steady flow (negligible inertial forces) and small Reynolds number, **(300 – 1)** can be simplified into the form

$$\eta \triangle \mathbf{v} - \nabla p = 0. \tag{300 – 6}$$

Considering the velocity vector of the fluid $\mathbf{u}$ at infinity, it could be substituted into the equation of continuity **(300 – 2)**

$$\mathrm{div}\,(\mathbf{v} - \mathbf{u}) = \mathrm{div}\,\mathbf{v} = 0, \tag{300 – 7}$$

$\mathbf{v} - \mathbf{u}$ can be expressed using vector potential $\mathbf{A}$, i.e. $\mathbf{rot}\,\mathbf{A} = 0$ at infinity

$$\mathbf{v} - \mathbf{u} = \mathbf{rot}\,\mathbf{A}, \tag{300 – 8}$$

this step can be justified by the Poincaré Lemma, [2]. The parity properties of vectors are used further. The velocity $\mathbf{v}$ is the real vector and the right-hand side of equation **(300 – 8)** has to be also the real vector. Therefore, $\mathbf{A}$ has to be a pseudovector. In the flow past a sphere (symmetrical body), the preferred direction is the direction of $\mathbf{u}$. The parameter $\mathbf{u}$ has to appear linearly in $\mathbf{A}$ due to the linear motion equations and boundary conditions. The general form of the vector function satisfying these requirements is

$$\mathbf{A} = f'(r)\mathbf{n} \times \mathbf{u}, \tag{300 – 9}$$

where $\mathbf{n}$ is a unit vector parallel to the position vector $\mathbf{r}$ and $f'(r)$ is a scalar function of r. The product $f'(r)\mathbf{n}$ can be expressed as the gradient of another function $f(r)$. Considering that $\mathbf{u}$ is constant, the velocity vector can be then written as

$$\begin{aligned} \mathbf{v} &= \mathbf{u} + \mathbf{rot}(\nabla f \times \mathbf{u}) \\ &= \mathbf{u} + \mathbf{rot}\,\mathbf{rot}(f\mathbf{u}). \end{aligned} \tag{300 – 10}$$

To determine the function f, the **rot** of **(300 – 6)** is used

$$\triangle \mathbf{rot}\, \mathbf{v} = 0. \tag{300 – 11}$$

The **rot** of the velocity field **v** is calculated at first

$$\begin{aligned} \mathbf{rot}\, \mathbf{v} &= \mathbf{rot}\, \mathbf{rot}\, \mathbf{rot}(f\mathbf{u}) \\ &= (\nabla \mathrm{div} - \triangle)\mathbf{rot}\,(f\mathbf{u})) \\ &= -\triangle\, \mathbf{rot}(f\mathbf{u}), \end{aligned} \tag{300 – 12}$$

and then substituted into **(300 – 11)** it takes the form

$$\begin{aligned} \triangle^2\, \mathbf{rot}(f\mathbf{u}) &= \triangle^2(\nabla \times \mathbf{u}) \\ &= (\triangle^2 \nabla f) \times \mathbf{u} = 0. \end{aligned} \tag{300 – 13}$$

From this it follows that

$$\triangle^2 \nabla f = 0. \tag{300 – 14}$$

A first integration of **(300 – 14)** gives

$$\triangle^2 f = \mathrm{constant} = 0. \tag{300 – 15}$$

The difference $\mathbf{v} - \mathbf{u}$ has to be zero at infinity, so the integration constant has to be equal to zero. Function f is the function of only one variable r, thus **(300 – 15)** can be written in the form

$$\triangle^2 f \equiv \frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2 \frac{\mathrm{d}}{\mathrm{d}r}\right) \triangle f = 0. \tag{300 – 16}$$

Solving this equation $\triangle f$ is equal to

$$\triangle f = \frac{2A}{r} + C, \tag{300 – 17}$$

where A and C are real constants. The constant C has to be zero due to the condition $\mathbf{v} - \mathbf{u} = 0$ at infinity. Further solution gives

$$f = Ar + \frac{B}{r}, \tag{300 – 18}$$

where B is a real constant. The additive constant is omitted because the velocity is given by the derivatives of f. After substitution of f into **(300 – 10)**,

$$\mathbf{v} = \mathbf{u} - A\frac{\mathbf{u} + \mathbf{n}(\mathbf{u}\cdot\mathbf{n})}{r} + B\frac{3\mathbf{n}(\mathbf{u}\cdot\mathbf{n}) - \mathbf{u}}{r^3}. \tag{300 – 19}$$

The constants A and B have to be determined from the boundary condition **(300 – 3)**. Hence

$$-\mathbf{u}\left(\frac{A}{R} + \frac{B}{R^3} - 1\right) + \mathbf{n}(\mathbf{u}\cdot\mathbf{n})\left(-\frac{A}{R} + \frac{3B}{R^3}\right) = 0. \tag{300 – 20}$$

The equation has to hold for all $\mathbf{n}$, thus

$$A = \frac{3}{4}R, \tag{300 – 21}$$

$$B = \frac{1}{4}R^3. \tag{300 – 22}$$

The final form of scalar function f and the velocity vector is obtained

$$f = \frac{3Rr}{4} + \frac{1R^3}{4r}, \tag{300 – 23}$$

$$\mathbf{v} = -\frac{3}{4}R\frac{\mathbf{u} + \mathbf{n}(\mathbf{u}\cdot\mathbf{n})}{r} + \frac{1}{4}R^3\frac{3\mathbf{n}(\mathbf{u}\cdot\mathbf{n}) - \mathbf{u}}{r^3} + \mathbf{u}. \tag{300 – 24}$$

The velocity vector components are determined in spherical polar coordinates with the axis parallel to $\mathbf{u}$

$$v_r = u\cos\theta\left(1 - \frac{3R}{2r} + \frac{R^3}{2r^3}\right), \tag{300 – 25}$$

$$v_\theta = -u\cos\theta\left(1 - \frac{3R}{4r} + \frac{R^3}{4r^3}\right). \tag{300 – 26}$$

This defines the velocity distribution around the sphere.

Pressure

To determine the pressure, **(300 – 10)** is substituted into **(300 – 6)**

$$\begin{aligned}\nabla p &= \eta \triangle \mathbf{v}\\ &= \eta \triangle \textbf{rot rot}(f\mathbf{u})\\ &= \eta \triangle (\nabla \mathrm{div}(f\mathbf{u})\mathbf{u} \triangle f).\end{aligned} \tag{300 – 27}$$

According to **(300 – 15)**, $\triangle^2 f = 0$ so it can be written that

$$\begin{aligned}\nabla p &= \nabla\left[\eta \triangle \operatorname{div}(f\mathbf{u})\right] \\ &= \nabla(\eta \mathbf{u} \cdot \nabla \triangle f). \end{aligned} \qquad (300-28)$$

Hence,

$$p = \eta \mathbf{u} \cdot \nabla \triangle f + p_0, \qquad (300-29)$$

where p_0 is the fluid pressure at infinity (in this case $p_0 = 0$). Substituting **(300 – 23)** for f, the final formula for pressure is obtained

$$p = p_0 - \frac{3}{2}\eta \frac{\mathbf{u} \cdot \mathbf{n}}{r^2} R. \qquad (300-30)$$

Drag Force

Now it is possible to determine the drag force on the sphere. The stress tensor σ can be divided into the hydrostatic and deviatoric part

$$\sigma = -p\mathbf{I} + \eta\left(\nabla \mathbf{v} + \nabla \mathbf{v}^{\mathsf{T}}\right). \qquad (300-31)$$

The force $\mathbf{F}$ per unit area with outer normal $\mathbf{n}$ is defined as

$$\mathbf{F} = -\sigma \mathbf{n}. \qquad (300-32)$$

The spherical polar coordinates with the axis parallel to $\mathbf{u}$ are considered and all quantities are functions only of r and of the polar angle θ. The drag force in the flow direction can be determined by means of integration appropriate forces per square area over the surface of the sphere

$$F_x = \int\limits_{\Omega} (-p \cos\theta + \sigma_{rr} \cos\theta - \sigma_{r\theta} \sin\theta) \mathrm{d}A, \qquad (300-33)$$

where the first term in the integral corresponds to the hydrostatic pressure, the second to the normal stress caused by the viscosity and third to the shear stress due to the viscosity. Appropriate stress components in spherical coordinates are following

$$\sigma_{rr} = 2\eta \frac{\partial v_r}{\partial r}, \qquad (300-34)$$

$$\sigma_{r\theta} = \eta\left(\frac{1}{r}\frac{\partial v_r}{\partial \theta} + \frac{\partial v_\theta}{\partial \theta} - \frac{v_\theta}{r}\right). \qquad (300-35)$$

Substituting expressions **(300 – 25)** and **(300 – 26)** it results

$$\sigma_{rr} = 0, \tag{300 – 36}$$

$$\sigma_{r\theta} = -\frac{3\eta}{2R} u \sin\theta. \tag{300 – 37}$$

The pressure p is defined by **(300 – 30)** and it can be expressed as

$$p = p_0 - \frac{3\eta}{2R} u \cos\theta. \tag{300 – 38}$$

Hence the integral **(300 – 33)** reduces to

$$F_x = 3\pi\mu Ru \int_0^\pi \cos^2\theta \sin\theta \mathrm{d}\theta + 3\pi\mu Ru \int_0^\pi \sin^3\theta \mathrm{d}\theta = \underbrace{2\pi\mu Ru}_{\text{pressure}} + \underbrace{4\pi\mu Ru}_{\textit{viscosity}}. \tag{300 – 39}$$

The total drag force of the sphere in the slow fluid flow (Stokes formula) follows

$$F_x = 2\pi\mu Ru + 4\pi\mu Ru = 6\pi\mu Ru. \tag{300 – 40}$$

In RWIND Simulation, viscous forces are neglected. Thus, the drag force is here calculated separately for pressure $F_{x,p}$ and viscous part $F_{x,v}$ according to **(300 – 40)**, and only the pressure part is further compared to the RWIND Simulation solution.

$$F_{x,p} = 2\pi\mu Ru \approx 3.770 \cdot 10^{-10}\mathrm{N} \tag{300 – 41}$$

$$F_{x,v} = 4\pi\mu Ru \approx 7.540 \cdot 10^{-10}\mathrm{N} \tag{300 – 42}$$

$$F_x = 6\pi\mu Ru \approx 1.131 \cdot 10^{-9}\mathrm{N} \tag{300 – 43}$$

RWIND Simulation Settings

- Modeled in RWIND Simulation 1.21
- Turbulence is not considered

Results

Structure Files	Program
0300.01	RWIND Simulation

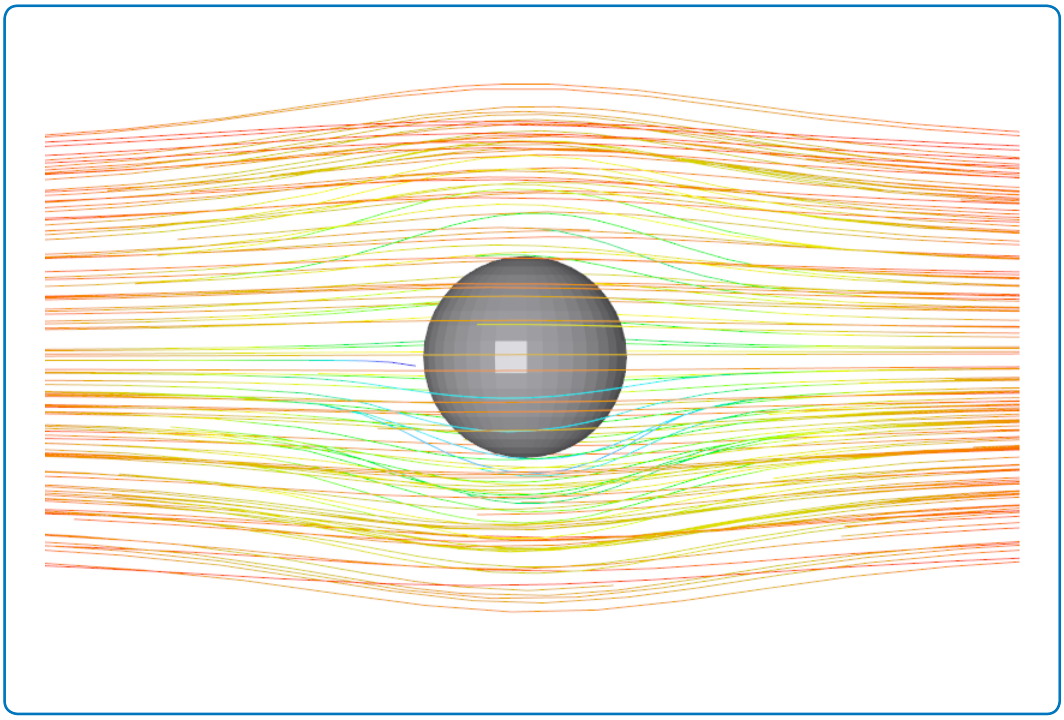

Figure 2: Results in RWIND Simulation - Streamlines

Analytical Solution	RWIND Simulation	
$F_{x,p}$ [N]	$F_{x,p}$ [N]	Ratio [-]
$3.770 \cdot 10^{-10}$	$3.637 \cdot 10^{-10}$	0.965

References

[1] LANDAU, L. and LIFSHITZ, E. *Fluid Mechanics*. Elsevier Science, 2013.

[2] SPIVAK, M. *Calculus on Manifolds: A Modern Approach to Classical Theorems of Advanced Calculus*. Avalon Publishing, 1965.