BACHELOR’S THESIS

International Civil Engineering B. Eng.

UNSYMMETRICAL BENDING

The Derivation of Constitutive Relationships for Analysis in the Displacement Method

Hochschule Mainz
University of Applied Sciences
Fachbereich Technik

Southern Illinois University
Edwardsville
Department of Civil Engineering

Handed in by	Benedikt Jugel
Matr. Nr.	943651
Student ID	800793260
Semester	8
Handed in to	Prof. Dr. Alaaeldin Elsisi Prof. Dr. Martin Neujahr
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Declaration

I hereby declare that the following bachelor's thesis,

"Unsymmetrical Bending – The Derivation of Constitutive Relationships for Analysis in the Displacement Method",

was created independently and without outside help. I only used the sources and tools specified in the work. The thoughts taken directly or indirectly from others are marked as such.

I also certify that I have not submitted or will submit either this or related work as an examination in other subjects.

Benedikt Jugel

B. Jugel

Edwardsville, 04/29/2024

Abstract

This thesis investigates how unsymmetrical bending can be expressed through the Displacement Method. It derives and solves ordinary systems of differential equations with real and complex solutions. In return, the solutions are used to derive closed-form constitutive relationships that describe unsymmetrical bending. The thesis focuses on deformations and internal forces of first and second order analysis. Furthermore, the relations are applied to exemplary systems and compared with the finite element software RSTAB 9. They are used to prove Euler's buckling load of a pinned-pinned column. This thesis expands upon existing theories for symmetrical bending and makes them applicable to unsymmetrical bending.

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Nomenclature

Symbol	Designation	Unit
$\vec{\boldsymbol{b}}$	Internal forces vector	m
$\vec{\boldsymbol{b}}_{\mathbf{0}}$	Force variables corresponding to the inhomogeneous solution	m
$\vec{\boldsymbol{b}}_{(e)}$	Element's force vector	m
c	Unknown constant of integration	–
du	Infinitesimal axial deformation	m
$d\varphi_y, d\varphi_z$	Infinitesimal rotation about the y-/z-axis	–
E	Modulus of Elasticity	N/m^2
$\boldsymbol{H}_{(e)}$	Element's Kinematic Relationship	–
i	Imaginary Unit	–
I_y, I_z	Second Moment of Area about the y-/z-axis	m^4
I_{yz}	Product Moment of Area	m^4
$\boldsymbol{K}$	Stiffness Matrix	N/m^3
$\boldsymbol{K}_{(e)}$	Element's Stiffness Matrix	N/m^3
L	Length of the beam	m
M_y, M_z	Bending Moment about the y-/z-axis	Nm
N	Normal Force	N
$N_{(f),i}$	Axial force of an arbitrary fiber	N
P	Axial compressive force	N
P_{cr}	Critical axial load, Buckling Load	N
Q	Substitute force that mirrors the effect Second Order Theory	N
q_y, q_z	Distributed load in y/z direction	N/m
$\vec{\boldsymbol{r}}$	Boundary force variables	N
$\vec{\boldsymbol{s}}$	Displacement vector, system's displacement vector	m
$\vec{\boldsymbol{s}}_{(e)}$	Element's displacement vector	m
S_y, S_z	First Moment of Area about the y-/z-axis	m^3
$\vec{\boldsymbol{u}}$	Boundary displacement variables	m

V_y, V_z	Shear Force in y/z direction	N
$\vec{\boldsymbol{w}}_{hom}$	Homogeneous solution	m
$\vec{\boldsymbol{w}}_{inhom}$	Inhomogeneous solution	m
w_y, w_z	Deflection in y/z direction	m
w'_y, w'_z	Tilt about the z-/y-axis	–
Δ	Determinant of the inertia tensor	m^8
$\Delta\mathrm{A}$	Area of a fiber	m^2
$\Delta L_{(f),i}$	Axial deformation of an arbitrary fiber	m
ε	Beam's stability identification value	–
ε_x	Uniform axial strain	–
$\vec{\boldsymbol{\eta}}$	Degree of Freedom	–
η	Amplitude functions of a deformation along the length of the beam	–
$\vec{\boldsymbol{\eta}}_{cr}$	Buckling Shape	–
κ	Stiffness Coefficients	–
λ	Stiffness Coefficients	–
μ	Amplitude functions of a bending moment along the length of the beam	–
Π_P	Potential of an axial load because of Second Order Theory	Nm
ρ_y, ρ_z	Standardized uniformly distributed load in y/z direction	–
φ_y, φ_z	Rotation about the y-/z-axis	–

1 Introduction

1.1 Motivation

Unsymmetrical Bending is a phenomenon in the bending analysis of beams. When a beam is loaded or analyzed in its non-principal axes, unsymmetrical bending occurs [1, p. 140]. While symmetrical bending of beams is covered extensively in most Mechanics classes and textbooks, unsymmetrical bending is poorly represented (see [1],[2]). When unsymmetrical bending is covered, generally only the calculations of stresses and strains are considered ([1],[3]). A formulation of relations between the forces and displacements that constitute unsymmetrical bending is rarely made. [1, p. 143] does give a simplified system of differential equations for when the bending moments of the beam are known. However, an expression of unsymmetrical bending for both statically determinate and indeterminate systems is difficult to find in the literature.

1.2 Objective

The objective of this thesis is to derive closed-form mathematical relations to be able to analyze beams subjected to unsymmetrical bending using the Displacement Method [4]. In the scope of this work, the relations are to be derived for the more general case of a beam, in which it is not described in its principal axes, but its non-principal axes. The reason for this decision is based on the argument that an analysis using a beam's principal axes may result in tedious or complex transformations. This idea is illustrated in the following figure.

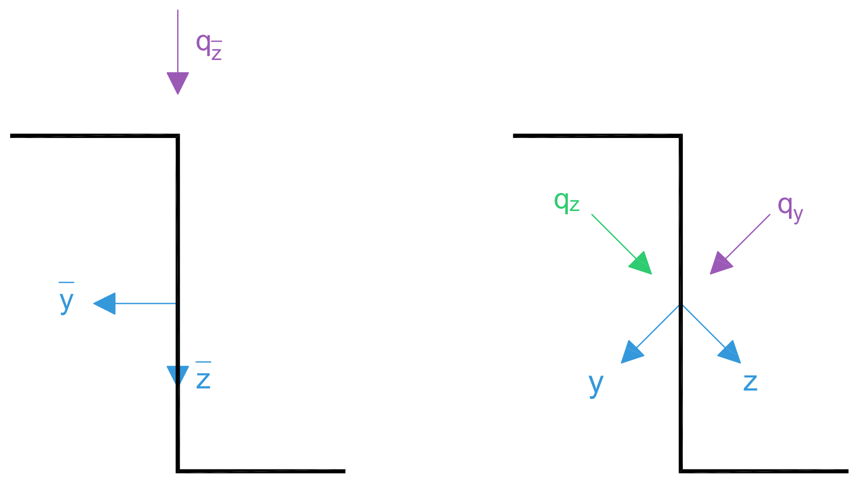


Fig. 1-1 Z-Section with non-principal axes (left) and principal axes (right) [5]

A beam with the shown cross-section is subject to a distributed load q_z that acts parallel to the non-principal z-axis. To describe the beam in its principal axes ($\hat{y}$ and $\hat{z}$), the section properties will first have to be determined with respect to the non-principal axes y and z. Then, the principal section properties can be determined. Additionally, the distributed load must be broken down into components $q_{\hat{z}}$ and $q_{\hat{y}}$ that act parallel to the principal axes. An analysis in the beam's non-principal axes would not feature any transformations of the section properties or load. In general, an analysis in a beam's non-principal axes can in certain situations negate these and further types of tedious transformations.

Moreover, for the beam shown below, it is more intuitive to analyze the beam with respect to its non-principal axes. The kinematic and equilibrium boundary conditions are simple when observed in the non-principal axes. An analysis in the principal axes would require transformations.

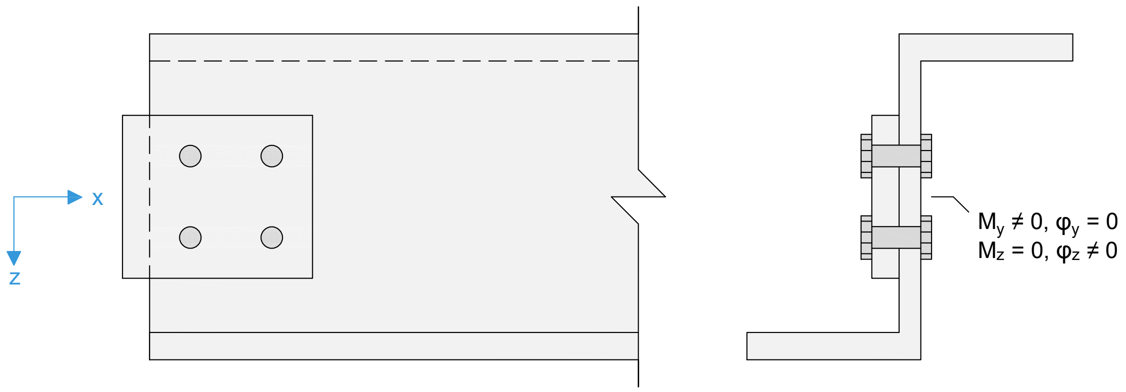


Fig. 1-2 Uniaxial Moment Connection of a Z-Beam [5]

This thesis focuses on constitutive relationships based on First Order Analysis and Second Order Analysis. These constitutive relationships yield relations between the internal forces and the displacements. As previously explained, these variables are described with respect to the beam's non-principal axes.

1.3 Methodology

In this thesis, a beam is treated as an element that fulfills a system of differential equations. Specifically, its deformations, expressed through the vector $\vec{w}$, must satisfy a specific system of differential equations. The solution of said system consists of homogeneous and inhomogeneous solutions.

$$\vec{w} = \vec{w}_{hom} + \vec{w}_{inhom} \tag{1.1}$$

With

$\vec{w}_{hom}$ ... Homogeneous Solution

$\vec{w}_{inhom}$ ... Inhomogeneous Solution

Furthermore, solving the system of differential equations provides relationships between the displacement and force variables at the element's boundaries with the constants of integration. These consist of expressions corresponding to the homogeneous and inhomogeneous solutions as well.

$$\vec{u} = \vec{u}_{hom} + \vec{u}_{inhom} \tag{1.2}$$

$$\vec{r} = \vec{r}_{hom} + \vec{r}_{inhom} \tag{1.3}$$

With

$\vec{u}$ ... Boundary displacement variables

$\vec{r}$ ... Boundary force variables

The homogeneous solutions of $\vec{u}$ and $\vec{r}$ can be rewritten using matrices, that relate the solution to the constants of integration. This results in:

$$\vec{u} = \boldsymbol{U} \cdot \vec{c} + \vec{u}_{inhom} \tag{1.4}$$

$$\vec{r} = \boldsymbol{R} \cdot \vec{c} + \vec{r}_{inhom} \tag{1.5}$$

Within the Displacement Method, certain adjustments are made to the boundary conditions. These can be expressed through the following matrix multiplications.

$$\vec{s} = \boldsymbol{S} \cdot \vec{u} \tag{1.6}$$

$$\vec{b} = \boldsymbol{B} \cdot \vec{r} \tag{1.7}$$

With

$\vec{s}$ ... Displacement vector

$\vec{b}$ ... Internal forces vector

Combing these expressions gives:

$$\vec{s} = \boldsymbol{S} \cdot \boldsymbol{U} \cdot \vec{c} + \boldsymbol{S} \cdot \vec{u}_{inhom} \tag{1.8}$$

$$\vec{b} = \boldsymbol{B} \cdot \boldsymbol{R} \cdot \vec{r} + \boldsymbol{B} \cdot \vec{r}_{inhom} \tag{1.9}$$

A connection between the displacement and force variables emerges through the rearrangement of (1.8) for $\vec{c}$, and the subsequent insertion of (1.10) in (1.11).

$$\vec{c} = \boldsymbol{U}^{-1} \cdot \boldsymbol{S}^{-1} \cdot \vec{s} - \boldsymbol{U}^{-1} \cdot \vec{u}_{inhom} \tag{1.10}$$

$$\vec{b} = \boldsymbol{B} \cdot \boldsymbol{R} \cdot \boldsymbol{U}^{-1} \cdot \boldsymbol{S}^{-1} \cdot \vec{s} - \boldsymbol{B} \cdot \boldsymbol{R} \cdot \boldsymbol{U}^{-1} \cdot \vec{u}_{inhom} + \boldsymbol{B} \cdot \vec{r}_{inhom} \tag{1.11}$$

The connection from (1.11) is the constitutive relationship of the beam element. It can be reduced to the following form:

$$\vec{\boldsymbol{b}} = \boldsymbol{K} \cdot \vec{\boldsymbol{s}} + \vec{\boldsymbol{b}}_{\boldsymbol{0}} \tag{1.12}$$

With

$\boldsymbol{K}$ ... Stiffness matrix

$\vec{\boldsymbol{b}}_{\boldsymbol{0}}$ ... Force variables corresponding to the inhomogeneous solutions

Where

$$\boldsymbol{K} := \boldsymbol{B} \cdot \boldsymbol{R} \cdot \boldsymbol{U}^{-1} \cdot \boldsymbol{S}^{-1} \tag{1.13}$$

$$\vec{\boldsymbol{b}}_{\boldsymbol{0}} := \boldsymbol{B} \cdot \vec{\boldsymbol{r}}_{\boldsymbol{inhom}} - \boldsymbol{B} \cdot \boldsymbol{R} \cdot \boldsymbol{U}^{-1} \cdot \vec{\boldsymbol{u}}_{\boldsymbol{inhom}} \tag{1.14}$$

The constitutive relationships can be used to analyze systems of with the Displacement Method. First, a kinematic relationship is set up, which relates the boundary displacements of each element with a system's deformations, also known as degrees of freedom.

$$\vec{\boldsymbol{s}}_{(e)} = \boldsymbol{H}_{(e)} \cdot \vec{\boldsymbol{s}} \tag{1.15}$$

With

$\vec{\boldsymbol{s}}_{(e)}$ ... Element's displacement vector

$\boldsymbol{H}_{(e)}$ ... Element's kinematic relationship matrix

$\vec{\boldsymbol{s}}$ ... System's displacement vector

For each element, a constitutive relationship of the form (1.12) is needed.

$$\vec{\boldsymbol{b}}_{(e)} = \boldsymbol{K}_{(e)} \cdot \vec{\boldsymbol{s}}_{(e)} \tag{1.16}$$

With

$\vec{\boldsymbol{b}}_{(e)}$ ... Element's force vector

$\boldsymbol{K}_{(e)}$ ... Element's stiffness matrix

Based on the forces acting on the system and the internal forces of each member, the equilibrium condition follows to:

$$\vec{\boldsymbol{b}} = \sum \boldsymbol{H}_{(e)}^{T} \cdot \vec{\boldsymbol{b}}_{(e)} \tag{1.17}$$

Inserting (1.15) and (1.16) in (1.17) yields the system equation:

$$\vec{\boldsymbol{b}} = \sum \boldsymbol{H}_{(e)}^{T} \cdot \boldsymbol{K}_{(e)} \cdot \boldsymbol{H}_{(e)} \cdot \vec{\boldsymbol{s}} = \boldsymbol{K} \cdot \vec{\boldsymbol{s}} \tag{1.18}$$

The system equation can then be used to determine the displacement vector $\vec{\boldsymbol{s}}$, which is used to calculate the internal forces of each member.

Through (1.15) and (1.16):

$$\vec{\boldsymbol{b}}_{(e)} = \boldsymbol{K}_{(e)} \cdot \boldsymbol{H}_{(e)} \cdot \vec{\boldsymbol{s}} \tag{1.19}$$

1.4 Exemplary Analyses

As a part of this thesis, several exemplary analyses are made. These primarily consist of analyzing an arbitrary system to demonstrate and apply the derived constitutive relationships. Additionally, any results that arise from the analyses are commented upon and discussed. A secondary function of these analyses is to compare the results with those of a structural analysis software. This provides a comparison of accuracy and checks the legitimacy of the derived constitutive relationships. The selected structural analysis software is the program “RSTAB 9 Structural Frame and Truss Analysis Software” by the company “Dlubal Software GmbH” (see [6]). It is a powerful 3D structural frame analysis software designed for 2D and 3D beam structures. The software uses a beam element approach, which is a specific application of the Finite Element Method [6].

2 Fundamentals

2.1 Order of Theories

There are three order of theories that describe the kinematics of a system. These are presented for a pinned column in the following figure.

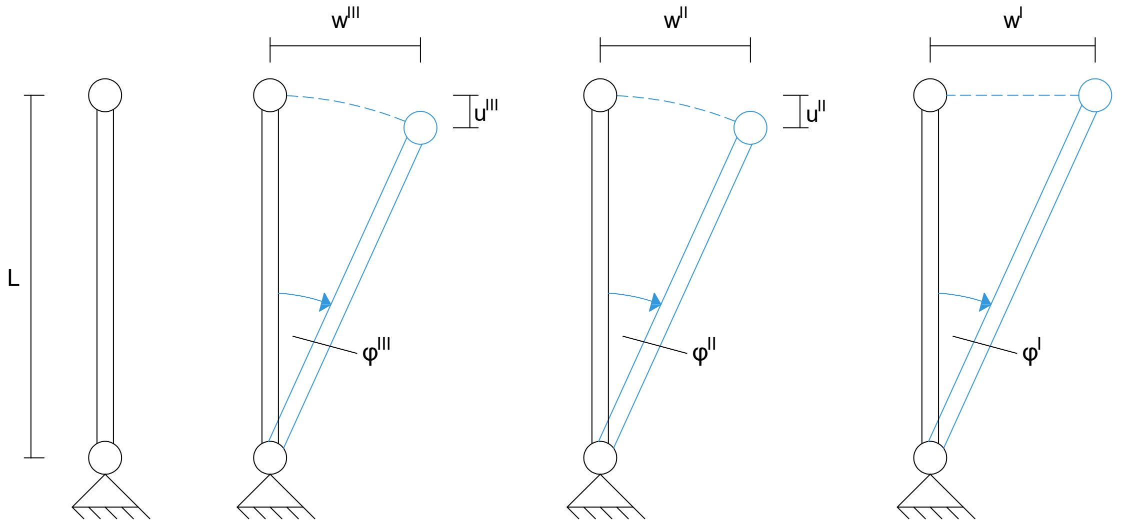


Fig. 2-1 Order of Theories illustrated for a Pinned Column [5]

Third Order Theory involves the exact deformation of the system, which here reflects a circular path made by the column head. Second Order and First Order Theory are approximations of Third Order Theory. The difference between Second and First Order Theory is that according to Second Order Theory a vertical deflection of the column head is considered.

From Fig. 2-1, the following kinematic relations can be acquired.

$$w^{III} = \sin(\varphi^{III}) \cdot L \tag{2.1}$$

$$w^{II} = \varphi^{II} \cdot L \tag{2.2}$$

$$w^{I} = \varphi^{I} \cdot L \tag{2.3}$$

$$u^{III} = [1 - \cos(\varphi^{III})] \cdot L \tag{2.4}$$

$$u^{II} = \frac{1}{2} \cdot (\varphi^{II})^2 \cdot L \tag{2.5}$$

$$u^{I} = 0 \tag{2.6}$$

Moreover, First Order Theory represents the concept of finding an equilibrium condition on the undeformed system. Conversely, Second and Third Order Theory follow the principle of finding an equilibrium on the deformed system. This is illustrated in the following figure.

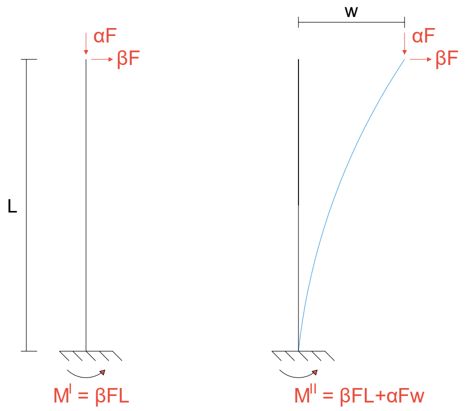


Fig. 2-2 Comparison of Equilibrium Conditions [5]

2.2 Deformations

The deformations of an infinitesimal beam element are illustrated in the following figure. With the assumption of small deformations, First Order Theory is applicable, and no axial displacement is considered for when the beam deflects in y or z-direction. According to the implemented coordinate system, positive deflections w_y and w_z are defined to be parallel to and in the direction of the y and z axes. The tilts of the beam, w_z' and w_y', are defined as the first derivatives of w_y and w_z with respect to x, respectively. The rotations φ_y and φ_z are defined by the right-hand rule, in which the thumb represents the axis of rotation, and the curl of the fingers represent the sense of rotation.

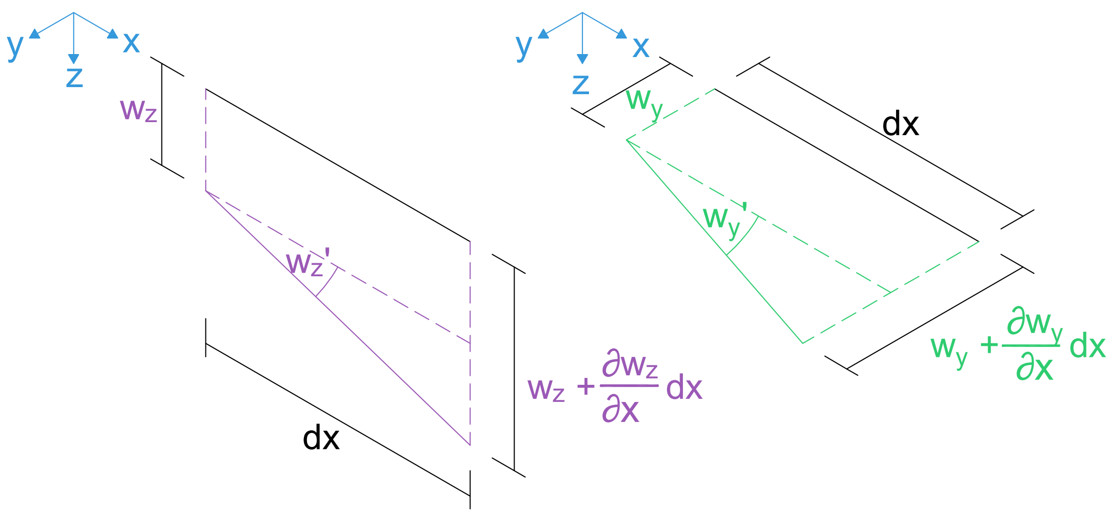


Fig. 2-3 Deformations of a Beam [5]

With

w_y ... Deflection in y direction

w_z ... Deflection in z direction

w_z' ... Tilt about the y-axis

w_y' ... Tilt about the z-axis

φ_y ... Rotation about the y-axis

φ_z ... Rotation about the z-axis

Where

$$\begin{bmatrix} \varphi_y \\ \varphi_z \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} w_z' \\ w_y' \end{bmatrix} \tag{2.7}$$

2.3 Internal Forces

The internal forces of an infinitesimal beam element are depicted in the following figure. The beam may experience a normal Force N, two shear forces V_z and V_y, and two bending moments M_y and M_y. Further forces, from torsion for example, are not considered within this thesis.

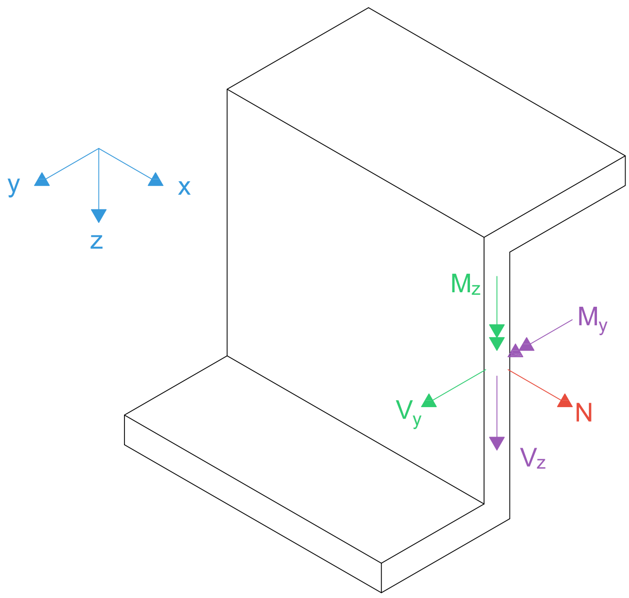


Fig. 2-4 Internal Forces of a Beam [5]

With

N ... Normal force

V_y, V_z ... Shear force in y/z direction

M_y, M_z ... Bending moment about the y/z-axis

The positive cut is defined as the cut where the normal force acts parallel to the x-axis. The negative cut is henceforth defined as the cut where the normal force acts antiparallel to the x-axis. At the positive cut, the shear forces act parallel to the y and z axes. The bending moments at the positive cut act parallel to the rotations about the y and z axes.

2.4 Modelling of the Beam

2.4.1 Beam Fiber Model

In the scope of this thesis, a beam is treated per the Bernoulli Hypothesis, that the plane sections remain plane and normal to the beam's longitudinal axis. The beam itself is modeled as a set of elastic fibers that can deform axially and may sway freely. The fibers are treated to only deform in their elastic state. Plastic analysis is not within the scope of this thesis. The beam fiber model is depicted below. The fibers possess the Modulus of Elasticity, E, of the beam and a cross-sectional area, ΔA. At the cross-section act the normal force and bending moments. The shear forces can be neglected since no shear deformations are assumed. An arbitrary point on the cross-section, D, is chosen as the reference point for the coordinate system, denoted by $\overline{xyz}$.

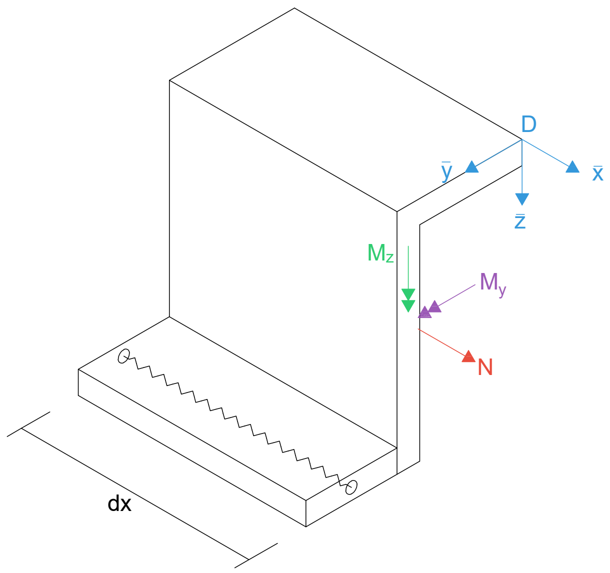


Fig. 2-5 Beam Fiber Model [5]

With

E ... Modulus of Elasticity

ΔA ... Area of a fiber

The beam element is herein modeled with three distinct degrees of freedom, $\vec{\boldsymbol{\eta}}_i$. The first degree of freedom represents the axial deformation of the beam. The second and third degrees of freedom represent the rotation of the cross-sectional plane about the y and z-axis, respectively. This is illustrated below.

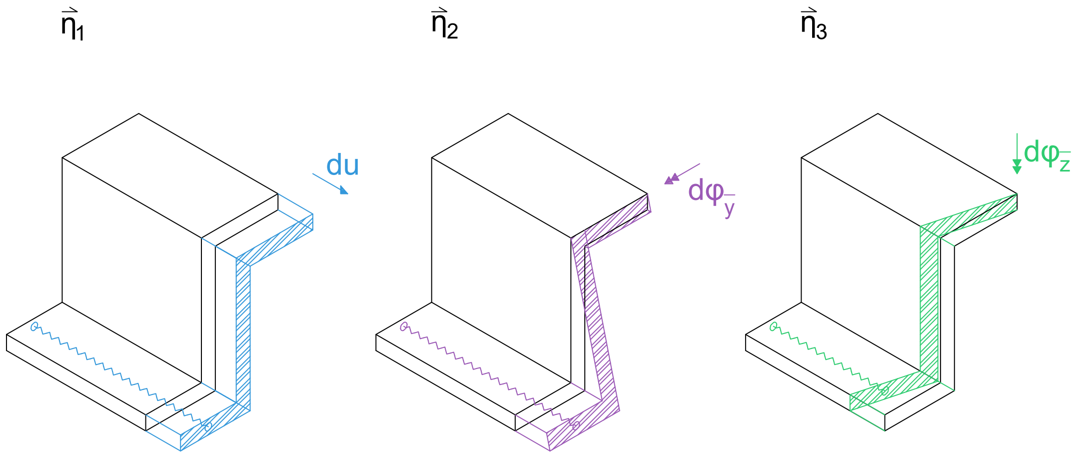


Fig. 2-6 Degrees of Freedom of the Cross-Section [5]

With

$\vec{\boldsymbol{\eta}}$	...	Degree of freedom
du	...	Infinitesimal axial deformation
$d\varphi_{\bar{y}}$	...	Infinitesimal rotation about the $\bar{y}$-axis
$d\varphi_{\bar{z}}$	...	Infinitesimal rotation about the $\bar{z}$-axis

Within each degree of freedom, the fibers undergo an axial strain relative to their position. For an arbitrary fiber, the kinematic relationship is:

$$\Delta L_{(f),i} = [1 \quad \bar{z} \quad -\bar{y}]_i \begin{bmatrix} du \\ d\varphi_{\bar{y}} \\ d\varphi_{\bar{z}} \end{bmatrix} \tag{2.8}$$

With

$\Delta L_{(f),i}$	...	Axial deformation of an arbitrary fiber

The fibers follow Hooke's Law and have the following constitutive relationship.

$$N_{(f),i} = \frac{E \cdot \Delta A}{dx} \cdot \Delta L_{(f),i} \tag{2.9}$$

With

$N_{(f),i}$	...	Axial force of an arbitrary fiber

The internal forces of the beam and all axial fiber forces must satisfy the following equilibrium conditions.

$$\begin{bmatrix} N \\ M_{\bar{y}} \\ M_{\bar{z}} \end{bmatrix} = \sum_{i=1}^{n} \begin{bmatrix} 1 \\ \bar{z} \\ -\bar{y} \end{bmatrix}_i \cdot N_{(f),i} \tag{2.10}$$

Combining (2.8) and (2.9) in (2.10) yields:

$$\begin{bmatrix} N \\ M_{\bar{y}} \\ M_{\bar{z}} \end{bmatrix} = \sum_{i=1}^{n} \begin{bmatrix} 1 \\ \bar{z} \\ -\bar{y} \end{bmatrix} \cdot \frac{E \cdot \Delta A}{dx} \cdot [1 \quad \bar{z} \quad -\bar{y}]_i \begin{bmatrix} du \\ d\varphi_{\bar{y}} \\ d\varphi_{\bar{z}} \end{bmatrix} \tag{2.11}$$

$$= \sum_{i=1}^{n} E \cdot \begin{bmatrix} 1 & \bar{z} & -\bar{z} \\ \bar{z} & \bar{z}^2 & -\bar{y}\bar{z} \\ -\bar{y} & -\bar{y}\bar{z} & \bar{y}^2 \end{bmatrix} \cdot \begin{bmatrix} du/dx \\ d\varphi_{\bar{y}}/dx \\ d\varphi_{\bar{z}}/dx \end{bmatrix} \cdot \Delta A \tag{2.12}$$

With the definition of a uniform strain $\varepsilon_{\bar{x}}$ and curvatures $\kappa_{\bar{y}}$ and $\kappa_{\bar{z}}$, the equation becomes simpler. Furthermore, the beam fiber model is only an accurate description for a set of infinitely many, but infinitely small fibers. That is:

$$\begin{bmatrix} N \\ M_{\bar{y}} \\ M_{\bar{z}} \end{bmatrix} = \lim_{n\to\infty} \lim_{\Delta A \to 0} \sum_{i=1}^{n} E \cdot \begin{bmatrix} 1 & \bar{z} & -\bar{z} \\ \bar{z} & \bar{z}^2 & -\bar{y}\bar{z} \\ -\bar{y} & -\bar{y}\bar{z} & \bar{y}^2 \end{bmatrix} \cdot \begin{bmatrix} \varepsilon_{\bar{x}} \\ \kappa_{\bar{y}} \\ \kappa_{\bar{z}} \end{bmatrix} \cdot \Delta A \tag{2.13}$$

With

$\varepsilon_{\bar{x}}$ ... Uniform axial strain

$\kappa_{\bar{y}}, \kappa_{\bar{z}}$ ... Curvatures

Where

$$\varepsilon_{\bar{x}} := \frac{du}{dx} = \frac{\partial u}{\partial x} \tag{2.14}$$

$$\kappa_{\bar{y}} := \frac{d\varphi_{\bar{y}}}{dx} = \frac{\partial \varphi_{\bar{y}}}{\partial x} \tag{2.15}$$

$$\kappa_{\bar{z}} := \frac{d\varphi_{\bar{z}}}{dx} = \frac{\partial \varphi_{\bar{z}}}{\partial x} \tag{2.16}$$

The summation resembles a Riemann Sum and can be rewritten using the integral sign.

$$\begin{bmatrix} N \\ M_{\bar{y}} \\ M_{\bar{z}} \end{bmatrix} = \int_0^A E \begin{bmatrix} 1 & \bar{z} & -\bar{z} \\ \bar{z} & \bar{z}^2 & -\bar{y}\bar{z} \\ -\bar{y} & -\bar{y}\bar{z} & \bar{y}^2 \end{bmatrix} \begin{bmatrix} \varepsilon_{\bar{x}} \\ \kappa_{\bar{y}} \\ \kappa_{\bar{z}} \end{bmatrix} \cdot dA \tag{2.17}$$

Defining the integrals as section properties leads to the constitutive relationship of the cross-section of the beam.

$$\begin{bmatrix} N \\ M_{\bar{y}} \\ M_{\bar{z}} \end{bmatrix} = E \begin{bmatrix} A & S_{\bar{y}} & -S_{\bar{z}} \\ S_{\bar{y}} & I_{\bar{y}} & I_{\overline{yz}} \\ -S_{\bar{z}} & I_{\overline{yz}} & I_{\bar{z}} \end{bmatrix} \begin{bmatrix} \varepsilon_{\bar{x}} \\ \kappa_{\bar{y}} \\ \kappa_{\bar{z}} \end{bmatrix} \tag{2.18}$$

With

$S_{\bar{y}}, S_{\bar{z}}$ ... First moment of area about the y-/z-axis

$I_{\bar{y}}, I_{\bar{z}}$ ... Second moment of area about the y-/z-axis

$I_{\overline{yz}}$ ... Product moment of area

Where

$$S_y := \int_0^A z \, dA \tag{2.19}$$

$$S_z := \int_0^A y \, dA \tag{2.20}$$

$$I_y := \int_0^A z^2 \, dA \tag{2.21}$$

$$I_z := \int_0^A y^2 \, dA \tag{2.22}$$

$$I_{yz} := -\int_0^A yz \, dA \tag{2.23}$$

2.4.2 Moment-Curvature Relationship

Within the scope of this thesis, the beam is analyzed with respect to its center of gravity and its non-principal axes. The section properties are thusly denoted by indices for the xyz coordinate system. This results in the first moments of area being equal to zero. Thus, bending and axial deformation occur independently of one another. This results in the moment-curvature relationship.

$$\begin{bmatrix} M_y \\ M_z \end{bmatrix} = E \cdot \begin{bmatrix} I_y & I_{yz} \\ I_{yz} & I_z \end{bmatrix} \cdot \begin{bmatrix} \kappa_y \\ \kappa_z \end{bmatrix} \tag{2.24}$$

3 First Order Analysis

3.1 System of Differential Equations

3.1.1 Derivation

A beam element with the following internal and external forces is considered. After an infinitesimal step in x direction, dx, the internal forces of the beam experience a change corresponding to their derivative with respect to x, $\frac{\partial}{\partial x}$, multiplied by dx. Moreover, the beam may be subjected to an arbitrarily distributed load q in either z or y direction. When observing an infinitesimal beam element, as in the figure below, these distributed loads act as a resultant force by being multiplied by dx.

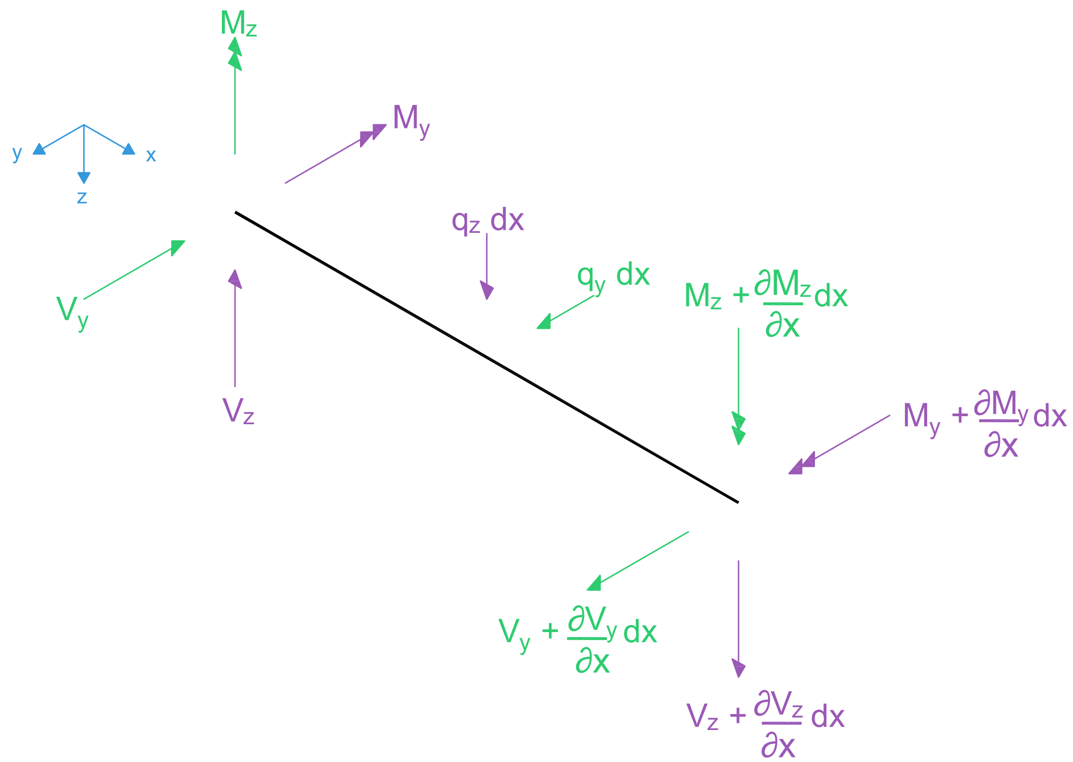


Fig. 3-1 Infinitesimal Beam Element with Internal and External Forces [5]

The equilibrium of forces in z and y direction yields:

$$\frac{\partial}{\partial x}\begin{bmatrix} V_z \\ V_y \end{bmatrix} = -\begin{bmatrix} q_z \\ q_y \end{bmatrix} \tag{3.1}$$

With

q_z, q_y ... Distributed load in z/y direction

The equilibrium of moments about the y and z axes gives:

$$\frac{\partial}{\partial x}\begin{bmatrix} M_y \\ M_z \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} V_z \\ V_y \end{bmatrix} \tag{3.2}$$

Differentiating (3.2) with respect to x allows (3.1) to be applied, which follows to:

$$\frac{\partial^2}{\partial x^2}\begin{bmatrix} M_y \\ M_z \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} q_z \\ q_y \end{bmatrix} \tag{3.3}$$

The moment curvature relationship, as expressed in (2.24) can be differentiated twice with respect to x. Within the scope of this thesis, the beam is assumed to have a uniform cross-section along its length. This criterion leads to:

$$\frac{\partial^2}{\partial x^2}\begin{bmatrix} M_y \\ M_z \end{bmatrix} = E \cdot \begin{bmatrix} I_y & I_{yz} \\ I_{yz} & I_z \end{bmatrix} \cdot \frac{\partial^2}{\partial x^2}\begin{bmatrix} \kappa_y \\ \kappa_z \end{bmatrix} \tag{3.4}$$

Using (2.7), (2.15) and (2.16), the curvatures take an expression related to the second derivatives of the deformations.

$$\begin{bmatrix} \kappa_y \\ \kappa_z \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\frac{\partial^2}{\partial x^2}\begin{bmatrix} w_z \\ w_y \end{bmatrix} \tag{3.5}$$

Combining (3.3), (3.4) and (3.5) gives the system of differential equations that describes a beam subjected to unsymmetrical bending.

$$E \cdot \begin{bmatrix} I_y & I_{yz} \\ I_{yz} & I_z \end{bmatrix} \cdot \frac{\partial^4}{\partial x^4}\begin{bmatrix} w_z \\ w_y \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} q_z \\ q_y \end{bmatrix} \tag{3.6}$$

Rewriting (3.6) gives the system of differential equations of the form $\frac{\partial^4}{\partial x^4}\vec{\boldsymbol{w}} = \boldsymbol{Q} \cdot \vec{\boldsymbol{q}}$.

$$\frac{\partial^4}{\partial x^4}\begin{bmatrix} w_z \\ w_y \end{bmatrix} = \frac{1}{E(I_y I_z - I_{yz}^2)}\begin{bmatrix} I_z & I_{yz} \\ I_{yz} & I_y \end{bmatrix}\begin{bmatrix} q_z \\ q_y \end{bmatrix} \tag{3.7}$$

The system of differential equations can be modified with the following definition of parameters.

$$\alpha := \frac{I_z}{I_y} \tag{3.8}$$

$$\omega := \frac{I_{yz}}{I_y} \tag{3.9}$$

Furthermore, the dimensionless coordinate ξ is introduced as:

$$\xi := \frac{x}{L} \tag{3.10}$$

With

L ... Length of the beam

The derivative with respect to x changes to the derivative with respect to ξ through:

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial \xi} \cdot \frac{\partial \xi}{\partial x} = \frac{\partial}{\partial \xi} \cdot \frac{1}{L} \tag{3.11}$$

Thus, the system of differential equations follows to:

$$\frac{\partial^4}{\partial \xi^4} \begin{bmatrix} w_z \\ w_y \end{bmatrix} = \frac{L^4}{EI_y(\alpha - \omega^2)} \begin{bmatrix} \alpha & \omega \\ \omega & 1 \end{bmatrix} \begin{bmatrix} q_z \\ q_y \end{bmatrix} \tag{3.12}$$

3.1.2 Solution

The homogeneous system of differential equations consists of two ordinary homogeneous linear fourth order differential equations.

$$\frac{\partial^4}{\partial \xi^4} \vec{\boldsymbol{w}} = \frac{\partial^4}{\partial \xi^4} \begin{bmatrix} w_z \\ w_y \end{bmatrix} = \vec{\boldsymbol{0}} \tag{3.13}$$

The solution to (3.13) may be expressed as a polynomial with unknown vectors of integration.

$$\vec{\boldsymbol{w}}_{hom} \cdot \frac{1}{L} = \vec{\boldsymbol{c}}_{\boldsymbol{1}} \cdot \xi^3 + \vec{\boldsymbol{c}}_{\boldsymbol{2}} \cdot \xi^2 + \vec{\boldsymbol{c}}_{\boldsymbol{3}} \cdot \xi + \vec{\boldsymbol{c}}_{\boldsymbol{4}} \tag{3.14}$$

Expanded, the homogeneous solution features eight distinct constants of integration, c.

$$\vec{\boldsymbol{w}}_{hom} \cdot \frac{1}{L} = \begin{bmatrix} c_{11} \\ c_{12} \end{bmatrix} \cdot \xi^3 + \begin{bmatrix} c_{21} \\ c_{22} \end{bmatrix} \cdot \xi^2 + \begin{bmatrix} c_{31} \\ c_{32} \end{bmatrix} \cdot \xi + \begin{bmatrix} c_{41} \\ c_{42} \end{bmatrix} \tag{3.15}$$

With

c ... Unknown constant of integration

In the scope of this thesis, the beam is assumed to be subject to uniformly distributed loads, denoted by q_{z0} and q_{y0}. The corresponding inhomogeneous system of ordinary linear fourth order differential equations is:

$$\frac{\partial^4}{\partial\xi^4}\begin{bmatrix} w_z \\ w_y \end{bmatrix} = \frac{L^4}{EI_y(\alpha-\omega^2)}\begin{bmatrix} \alpha & \omega \\ \omega & 1 \end{bmatrix}\begin{bmatrix} q_{z0} \\ q_{y0} \end{bmatrix} \tag{3.16}$$

The inhomogeneous solution is then:

$$\vec{\boldsymbol{w}}_{inhom} \cdot \frac{1}{L} = \frac{\xi^4}{24} \cdot \frac{1}{(\alpha-\omega^2)}\begin{bmatrix} \alpha & \omega \\ \omega & 1 \end{bmatrix}\begin{bmatrix} \rho_{z0} \\ \rho_{y0} \end{bmatrix} \tag{3.17}$$

With

ρ_{z0}, ρ_{y0} ... Standardized uniformly distributed loads

Where

$$\begin{bmatrix} \rho_{z0} \\ \rho_{y0} \end{bmatrix} := \frac{L^3}{EI_y}\begin{bmatrix} q_{z0} \\ q_{y0} \end{bmatrix} \tag{3.18}$$

The total solution of the system of differential equations for a beam subjected to Unsymmetrical bending is:

$$\vec{\boldsymbol{w}} = \vec{\boldsymbol{w}}_{hom} + \vec{\boldsymbol{w}}_{inhom} \tag{3.19}$$

Which follows to the following using the solutions presented in (3.15) and (3.17).

$$\vec{\boldsymbol{w}} \cdot \frac{1}{L} = \begin{bmatrix} c_{11} \\ c_{12} \end{bmatrix} \cdot \xi^3 + \begin{bmatrix} c_{21} \\ c_{22} \end{bmatrix} \cdot \xi^2 + \begin{bmatrix} c_{31} \\ c_{32} \end{bmatrix} \cdot \xi + \begin{bmatrix} c_{41} \\ c_{42} \end{bmatrix} + \frac{\xi^4}{24} \cdot \frac{1}{(\alpha-\omega^2)}\begin{bmatrix} \alpha & \omega \\ \omega & 1 \end{bmatrix}\begin{bmatrix} \rho_{z0} \\ \rho_{y0} \end{bmatrix} \tag{3.20}$$

3.1.3 Deformations

Using the solution presented in (3.20), the deformations can be expressed dependent on the constants of integration.

$$\begin{bmatrix} w_z \\ w_y \end{bmatrix} \cdot \frac{1}{L} = \begin{bmatrix} \xi^3 & \xi^2 & \xi & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \xi^3 & \xi^2 & \xi & 1 \end{bmatrix}\begin{bmatrix} c_{11} \\ c_{21} \\ c_{31} \\ c_{41} \\ c_{12} \\ c_{22} \\ c_{32} \\ c_{42} \end{bmatrix} + \frac{1}{\alpha-\omega^2}\begin{bmatrix} \frac{\alpha\xi^4}{24} & \frac{\omega\xi^4}{24} \\ \frac{\omega\xi^4}{24} & \frac{\xi^4}{24} \end{bmatrix}\begin{bmatrix} \rho_{z0} \\ \rho_{y0} \end{bmatrix} \tag{3.21}$$

Furthermore, the tilts are required. In 2.2, these were defined as the derivatives of the deformations w_z and w_y with respect to x. In accordance with (3.11), the derivative with respect to x is appropriately changed to with respect to ξ.

$$\begin{bmatrix} w_z' \\ w_y' \end{bmatrix} = \frac{\partial}{\partial x}\begin{bmatrix} w_z \\ w_y \end{bmatrix} = \frac{\partial}{\partial \xi}\begin{bmatrix} w_z \\ w_y \end{bmatrix} \cdot \frac{1}{L} \tag{3.22}$$

It follows that:

$$\begin{bmatrix} w_z' L \\ w_y' L \end{bmatrix} \cdot \frac{1}{L} = \begin{bmatrix} 3\xi^2 & 2\xi & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3\xi^2 & 2\xi & 1 & 0 \end{bmatrix} \begin{bmatrix} c_{11} \\ c_{21} \\ c_{31} \\ c_{41} \\ c_{12} \\ c_{22} \\ c_{32} \\ c_{42} \end{bmatrix} + \frac{1}{\alpha - \omega^2} \begin{bmatrix} \frac{\alpha\xi^3}{6} & \frac{\omega\xi^3}{6} \\ \frac{\omega\xi^3}{6} & \frac{\xi^3}{6} \end{bmatrix} \begin{bmatrix} \rho_{z0} \\ \rho_{y0} \end{bmatrix} \tag{3.23}$$

Letting ξ be equal to zero and one, grants the relationship between the boundary displacement variables and the constants of integration of the form $\vec{\boldsymbol{u}} = \boldsymbol{U} \cdot \vec{\boldsymbol{c}} + \vec{\boldsymbol{u}}_{inhom}$.

$$\begin{bmatrix} w_z(0) \\ w_z'(0)L \\ w_z(1) \\ w_z'(1)L \\ w_y(0) \\ w_y'(0)L \\ w_y(1) \\ w_y'(1)L \end{bmatrix} \cdot \frac{1}{L} = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 3 & 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 3 & 2 & 1 & 0 \end{bmatrix} \begin{bmatrix} c_{11} \\ c_{21} \\ c_{31} \\ c_{41} \\ c_{12} \\ c_{22} \\ c_{32} \\ c_{42} \end{bmatrix} + \frac{1}{\alpha - \omega^2} \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ \alpha/24 & \omega/24 \\ \alpha/6 & \omega/6 \\ 0 & 0 \\ 0 & 0 \\ \omega/24 & 1/24 \\ \omega/6 & 1/6 \end{bmatrix} \begin{bmatrix} \rho_{z0} \\ \rho_{y0} \end{bmatrix} \tag{3.24}$$

3.1.4 Internal Forces

The internal forces of the beam are, expressed through the moment-curvature relationship and the equilibrium conditions, dependent on the constants of integration. The parameterized moment-curvature relationship is:

$$\begin{bmatrix} M_y \\ M_z \end{bmatrix} = EI_y \cdot \begin{bmatrix} 1 & \omega \\ \omega & \alpha \end{bmatrix} \cdot \begin{bmatrix} \kappa_y \\ \kappa_z \end{bmatrix} \tag{3.25}$$

Using the expression for the curvatures in (3.5) gives:

$$\begin{bmatrix} M_y \\ M_z \end{bmatrix} = EI_y \cdot \begin{bmatrix} 1 & \omega \\ \omega & \alpha \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} w_z'' \\ w_y'' \end{bmatrix} \tag{3.26}$$

The second derivatives are hereby:

$$\begin{bmatrix} w_z'' \\ w_y'' \end{bmatrix} L = \begin{bmatrix} 6\xi & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 6\xi & 2 & 0 & 0 \end{bmatrix} \begin{bmatrix} c_{11} \\ c_{21} \\ c_{31} \\ c_{41} \\ c_{12} \\ c_{22} \\ c_{32} \\ c_{42} \end{bmatrix} + \frac{1}{\alpha - \omega^2} \begin{bmatrix} \frac{\alpha\xi^2}{2} & \frac{\omega\xi^2}{2} \\ \frac{\omega\xi^2}{2} & \frac{\xi^2}{2} \end{bmatrix} \begin{bmatrix} \rho_{z0} \\ \rho_{y0} \end{bmatrix} \tag{3.27}$$

Inserting (3.27) into (3.26) yields the moments-constants of integration relationship.

$$\begin{bmatrix} M_y \\ M_z \end{bmatrix} = \frac{EI_y}{L} \begin{bmatrix} -6\xi & -2 & 0 & 0 & 6\omega\xi & 2\omega & 0 & 0 \\ -6\omega\xi & -2\omega & 0 & 0 & 6\alpha\xi & 2\alpha & 0 & 0 \end{bmatrix} \begin{bmatrix} c_{11} \\ c_{21} \\ c_{31} \\ c_{41} \\ c_{12} \\ c_{22} \\ c_{32} \\ c_{42} \end{bmatrix} + \frac{EI_y}{L} \begin{bmatrix} \frac{-\xi^2}{2} & 0 \\ 0 & \frac{\xi^2}{2} \end{bmatrix} \begin{bmatrix} \rho_{z0} \\ \rho_{y0} \end{bmatrix} \tag{3.28}$$

The equilibrium condition shown in (3.2) can be inversed, which gives the shear forces as a function of the moments.

$$\begin{bmatrix} V_z \\ V_y \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \frac{\partial}{\partial x} \begin{bmatrix} M_y \\ M_z \end{bmatrix} \tag{3.29}$$

Applying this relation leads to the following:

$$\begin{bmatrix} V_z \\ V_y \end{bmatrix} = \frac{EI_y}{L^2} \begin{bmatrix} -6 & 0 & 0 & 0 & 6\omega & 0 & 0 & 0 \\ 6\omega & 0 & 0 & 0 & -6\alpha & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} c_{11} \\ c_{21} \\ c_{31} \\ c_{41} \\ c_{12} \\ c_{22} \\ c_{32} \\ c_{42} \end{bmatrix} + \frac{EI_y}{L^2} \begin{bmatrix} -\xi & 0 \\ 0 & -\xi \end{bmatrix} \begin{bmatrix} \rho_{z0} \\ \rho_{y0} \end{bmatrix} \tag{3.30}$$

Letting ξ be equal to zero and one, grants the relationship between the boundary force variables and the constants of integration. It takes the form $\vec{\boldsymbol{r}} = \boldsymbol{R} \cdot \vec{\boldsymbol{c}} + \vec{\boldsymbol{r}}_{inhom}$.

$$\begin{bmatrix} V_z(0) \\ M_y(0)/L \\ V_z(1) \\ M_y(1)/L \\ V_y(0) \\ M_z(0)/L \\ V_y(1) \\ M_z(1)/L \end{bmatrix} = \frac{EI_y}{L^2} \begin{bmatrix} -6 & 0 & 0 & 0 & 6\omega & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 & 0 & 2\omega & 0 & 0 \\ -6 & 0 & 0 & 0 & 6\omega & 0 & 0 & 0 \\ -6 & -2 & 0 & 0 & 6\omega & 2\omega & 0 & 0 \\ 6\omega & 0 & 0 & 0 & -6\alpha & 0 & 0 & 0 \\ 0 & -2\omega & 0 & 0 & 0 & 2\alpha & 0 & 0 \\ 6\omega & 0 & 0 & 0 & -6\alpha & 0 & 0 & 0 \\ -6\omega & -2\omega & 0 & 0 & 6\alpha & 2\alpha & 0 & 0 \end{bmatrix} \begin{bmatrix} c_{11} \\ c_{21} \\ c_{31} \\ c_{41} \\ c_{12} \\ c_{22} \\ c_{32} \\ c_{42} \end{bmatrix} + \frac{EI_y}{L^2} \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ -1 & 0 \\ -1/2 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & -1 \\ 0 & 1/2 \end{bmatrix} \begin{bmatrix} \rho_{z0} \\ \rho_{y0} \end{bmatrix} \tag{3.31}$$

3.2 Displacement Method

3.2.1 Definition of Displacements

Within the Displacement Method, the boundary deflections of the beam shall be parallel to the corresponding axes. Likewise, the boundary rotations rotate according to the right-hand rule around their respective axes. Illustrated in the following figure are the definitions of the displacements within the Displacement Method and of the system of differential equations.

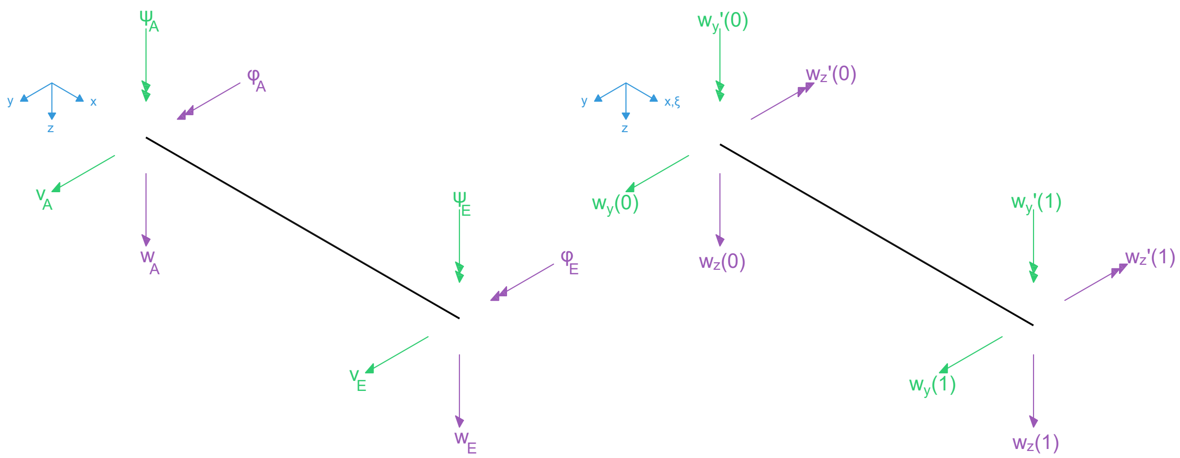


Fig. 3-2 Displacements of the Beam [5]

Comparing the two results in the relationship of the form $\vec{s} = \boldsymbol{S} \cdot \vec{\boldsymbol{u}}$.

$$\begin{bmatrix} w_A \\ \varphi_A L \\ w_E \\ \varphi_E L \\ v_A \\ \psi_A L \\ v_E \\ \psi_E L \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} w_z(0) \\ w_z'(0)L \\ w_z(1) \\ w_z'(1)L \\ w_y(0) \\ w_y'(0)L \\ w_y(1) \\ w_y'(1)L \end{bmatrix} \tag{3.32}$$

3.2.2 Definition of Forces

Within the Displacement Method, the boundary forces shall be equivalently defined as the boundary displacements. Illustrated in the following figure are the definitions within the Displacement Method and of the system of differential equations.

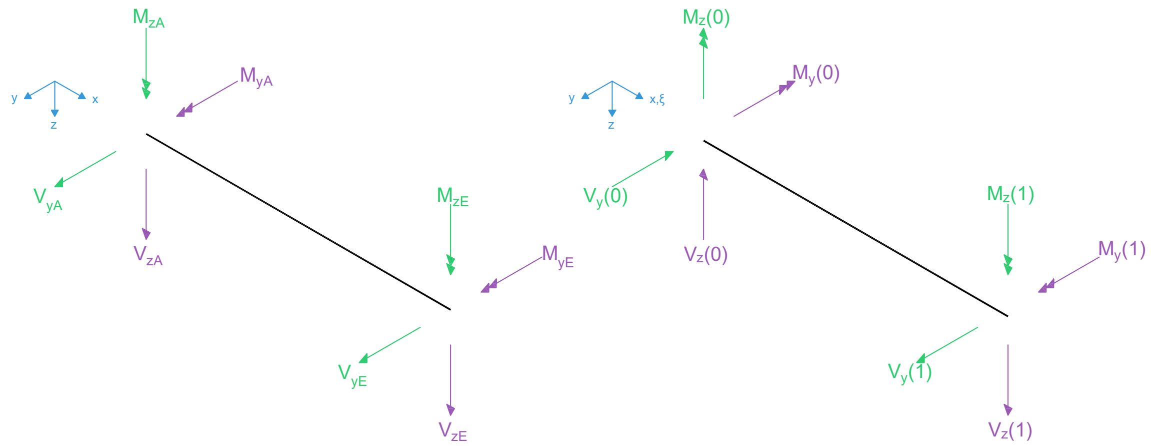


Fig. 3-3 Forces of the Beam [5]

Comparing the two results in the relationship of the form $\vec{\boldsymbol{b}} = \boldsymbol{B} \cdot \vec{\boldsymbol{r}}$.

$$\begin{bmatrix} V_{zA} \\ M_{yA}/L \\ V_{zE} \\ M_{yE}/L \\ V_{yA} \\ M_{zA}/L \\ V_{yE} \\ M_{zE}/L \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} V_z(0) \\ M_y(0)/L \\ V_z(1) \\ M_y(1)/L \\ V_y(0) \\ M_z(0)/L \\ V_y(1) \\ M_z(1)/L \end{bmatrix} \tag{3.33}$$

3.2.3 Constitutive Relationship

As described in 1.3, the constitutive relationship consists of the force vector $\vec{\boldsymbol{b}}$, displacement vector $\vec{\boldsymbol{s}}$, stiffness matrix $\boldsymbol{K}$ and a force vector $\vec{\boldsymbol{b}}_{\boldsymbol{0}}$, which corresponds to the inhomogeneous solution. The force vector and displacement vector are defined as:

$$\vec{\boldsymbol{b}} = \begin{bmatrix} V_{zA} \\ M_{yA}/L \\ V_{zE} \\ M_{yE}/L \\ V_{yA} \\ M_{zA}/L \\ V_{yE} \\ M_{zE}/L \end{bmatrix} \quad (3.34) \qquad \vec{\boldsymbol{s}} = \begin{bmatrix} w_A \\ \varphi_A L \\ w_E \\ \varphi_E L \\ v_A \\ \psi_A L \\ v_E \\ \psi_E L \end{bmatrix} \quad (3.35)$$

The stiffness matrix within this section is of First Order Theory and shall be denoted as $\boldsymbol{K^I}$. The stiffness matrix is determined by (1.13) with the matrices $\boldsymbol{B}$, $\boldsymbol{R}$, $\boldsymbol{U}$ and $\boldsymbol{S}$ originating from (3.33), (3.31), (3.24) and (3.32), respectively. It follows that:

$$\boldsymbol{K^I} = \frac{EI_y}{L^3} \begin{bmatrix} 12 & -6 & -12 & -6 & -12\omega & -6\omega & 12\omega & -6\omega \\ -6 & 4 & 6 & 2 & 6\omega & 4\omega & -6\omega & 2\omega \\ -12 & 6 & 12 & 6 & 12\omega & 6\omega & -12\omega & 6\omega \\ -6 & 2 & 6 & 4 & 6\omega & 2\omega & -6\omega & 4\omega \\ -12\omega & 6\omega & 12\omega & 6\omega & 12\alpha & 6\alpha & -12\alpha & 6\alpha \\ -6\omega & 4\omega & 6\omega & 2\omega & 6\alpha & 4\alpha & -6\alpha & 2\alpha \\ 12\omega & -6\omega & -12\omega & -6\omega & -12\alpha & -6\alpha & 12\alpha & 6\alpha \\ -6\omega & 2\omega & 6\omega & 4\omega & 6\alpha & 2\alpha & 6\alpha & 4\alpha \end{bmatrix} \quad (3.36)$$

The stiffness matrix is symmetric, that is $\boldsymbol{K} = \boldsymbol{K}^T$. It can be written using four submatrices.

$$\boldsymbol{K} = \begin{bmatrix} \boldsymbol{K}_y & \boldsymbol{K}_{yz} \\ \boldsymbol{K}_{yz}^T & \boldsymbol{K}_z \end{bmatrix} \quad (3.37)$$

This form highlights the fact that bending about the y and z axes is controlled by stiffness submatrices $\boldsymbol{K}_y$ and $\boldsymbol{K}_z$, respectively. The submatrix $\boldsymbol{K}_{yz}$ and its transposed $\boldsymbol{K}_{yz}^T$ are responsible for coupling the bending about each axis. Henceforth, $\boldsymbol{K}_{yz}$ is responsible for unsymmetrical bending. This also demonstrates the known fact, that for $I_{yz} = 0$, that is $\omega = 0$, unsymmetrical bending vanishes.

The force vector that corresponds to the inhomogeneous solution, $\vec{\boldsymbol{b}}_{\boldsymbol{0}}$, is of First Order Theory and shall be denoted by $\vec{\boldsymbol{b}}_0^I$. This vector is calculated through (1.14) with the matrices $\boldsymbol{B}$, $\boldsymbol{R}$, $\boldsymbol{U}$ and the vectors $\vec{\boldsymbol{r}}_{inhom}$ and $\vec{\boldsymbol{u}}_{inhom}$ from (3.33), (3.31), (3.24), (3.31) and (3.24), respectively. This leads to the following vector.

$$\vec{\boldsymbol{b}}_0^I = \frac{EI_y}{L^2} \begin{bmatrix} -1/2 & 0 \\ 1/12 & 0 \\ -1/2 & 0 \\ -1/12 & 0 \\ 0 & -1/2 \\ 0 & -1/12 \\ 0 & -1/2 \\ 0 & 1/12 \end{bmatrix} \begin{bmatrix} \rho_{z0} \\ \rho_{y0} \end{bmatrix} \tag{3.38}$$

Evidently, a distributed load acting in z direction will only produce forces in z direction and moments about the y axis. The same goes for a distributed load acting in y direction. This qualitative equilibrium condition is met with the above vector.

3.2.4 Deformations

A relation between the deformations $w_z(\xi)$ and $w_y(\xi)$ and the displacement vector can be made. Using (3.21), (3.24) and (3.32), the deformations are expressed through the amplitude functions η. Since they are based on First Order Theory, they are denoted as such.

$$\begin{bmatrix} w_z(\xi) \\ w_y(\xi) \end{bmatrix} = \begin{bmatrix} \eta_{w_A}^I & \eta_{\varphi_A}^I & \eta_{w_E}^I & \eta_{\varphi_E}^I & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \eta_{v_A}^I & \eta_{\psi_A}^I & \eta_{v_E}^I & \eta_{\psi_E}^I \end{bmatrix} \begin{bmatrix} w_A \\ \varphi_A L \\ w_E \\ \varphi_E L \\ v_A \\ \psi_A L \\ v_E \\ \psi_E L \end{bmatrix} + \frac{\eta_\rho^I}{24(\alpha - \omega^2)} \begin{bmatrix} \alpha & \omega \\ \omega & 1 \end{bmatrix} \begin{bmatrix} \rho_{z0} L \\ \rho_{y0} L \end{bmatrix} \tag{3.39}$$

With

η ... Amplitude function of a deformation along the length of the beam

The definitions of these amplitude functions are given in Attachment A. They represent the eight degrees of freedom of the beam, as well as the deformations induced by uniformly distributed loads. These are illustrated below. It can be seen, that due to a displacement w_z, only displacements along the z axis occur. Unsymmetrical bending does not occur, when a beam is subject to a boundary nodal deformation.

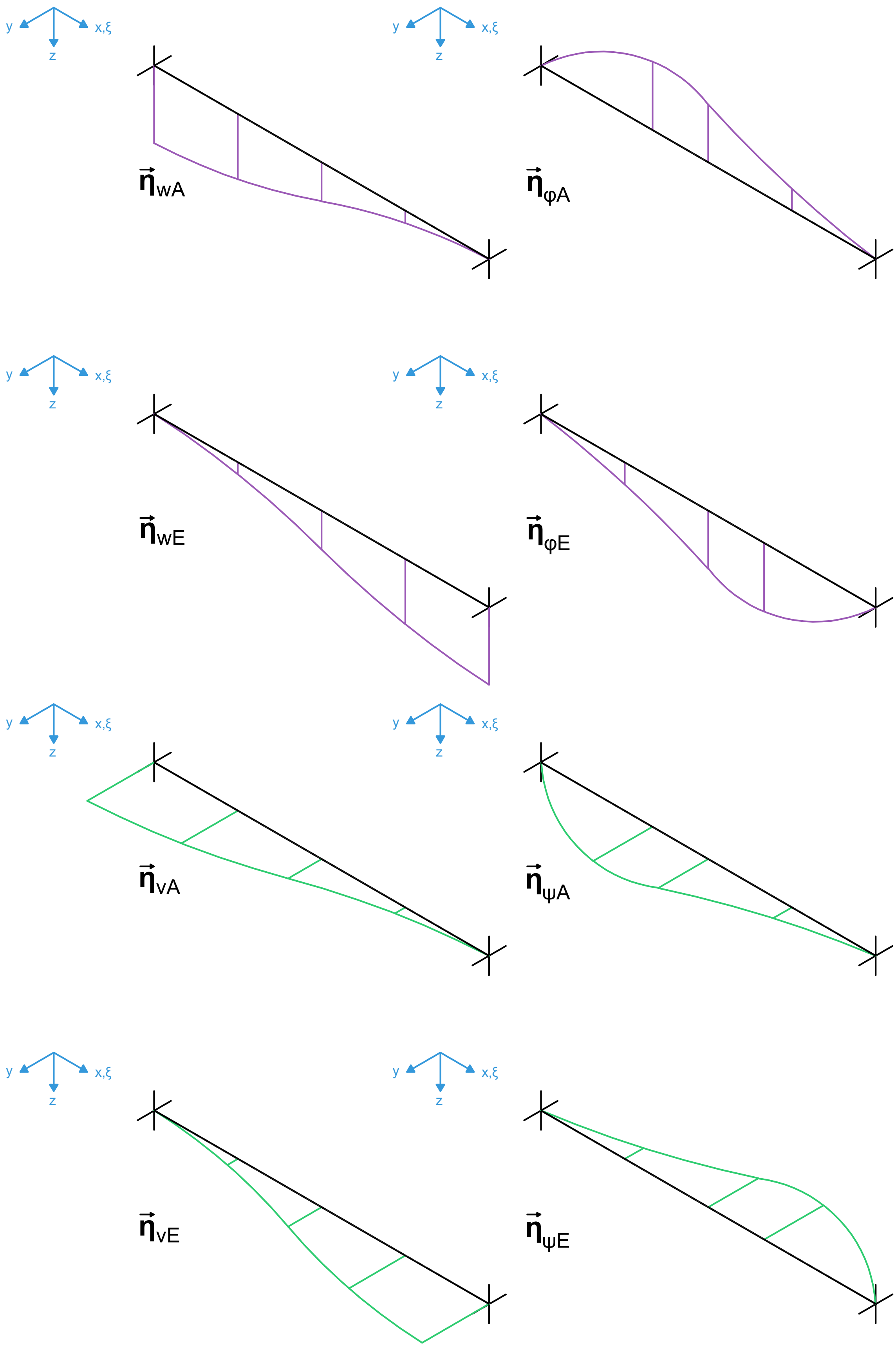


Fig. 3-4 First Order Deformation Amplitude Functions [5]

3.2.5 Internal Forces

To find the internal forces of the beam, a relation between the internal forces and the displacement vector is made. Using (3.24), (3.28) and (3.32), the bending moments of the beam can be expressed similarly to (3.39).

$$\begin{bmatrix} M_y \\ M_z \end{bmatrix} = \frac{EI_y}{L^2} \begin{bmatrix} \mu^I_{w_A} & \mu^I_{\varphi_A} & \mu^I_{w_E} & \mu^I_{\varphi_E} & \mu^I_{wv_A} & \mu^I_{\varphi\psi_A} & \mu^I_{wv_E} & \mu^I_{\varphi\psi_E} \\ -\mu^I_{wv_A} & \mu^I_{\varphi\psi_A} & -\mu^I_{wv_E} & \mu^I_{\varphi\psi_E} & \mu^I_{v_A} & \mu^I_{\psi_A} & \mu^I_{v_E} & \mu^I_{\psi_E} \end{bmatrix} \begin{bmatrix} w_A \\ \varphi_A L \\ w_E \\ \varphi_E L \\ v_A \\ \psi_A L \\ v_E \\ \psi_E L \end{bmatrix} + \frac{EI_y}{L} \mu^I_\rho \begin{bmatrix} \rho_{z0} \\ \rho_{y0} \end{bmatrix} \tag{3.40}$$

With

μ ... Amplitude function of a bending moment along the length of the beam

The definitions of these amplitude functions are given in Attachment B. The amplitude functions μ_{wv} and $\mu_{\varphi\psi}$ also represent the coupling of bending about both axes. They reaffirm that unsymmetrical bending vanishes when $\omega = 0$. Moreover, they show that due to a deformation about one axis, internal forces appear about both axes. This is illustrated in Fig. 3-5. This phenomenon is analogous to the scenario in which a cantilever is subject a bending moment about one axis. The moment causes deformations about two axes, whilst only acting in one. This is illustrated in Fig. 3-6.

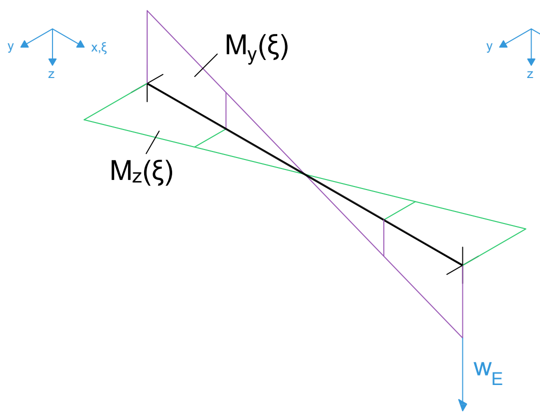


Fig. 3-5 Biaxial Moments caused by Uniaxial Deformation [5]

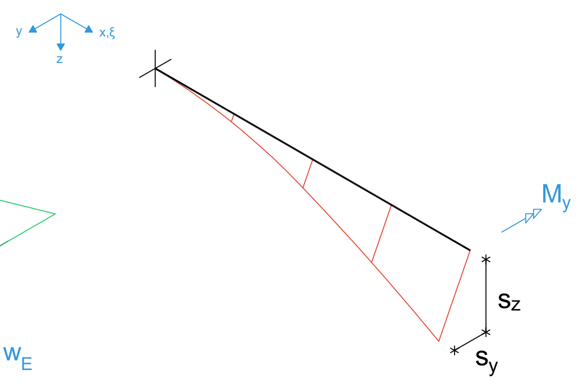


Fig. 3-6 Biaxial Deformation caused by Uniaxial Moment [5]

3.3 Exemplary Analyses

3.3.1 Cantilever

In the following scenario a cantilever beam is subjected to a load F at its free end as seen in the following figure.

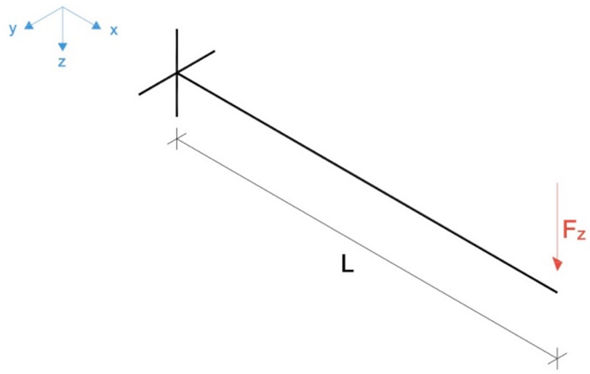


Fig. 3-7 Cantilever [5]

This beam has four degree of freedoms that are affected by the external load. The leading deformations in each degree of freedom shall be denoted by s_i. Illustrated below are the degrees of freedom and their respective leading deformations.

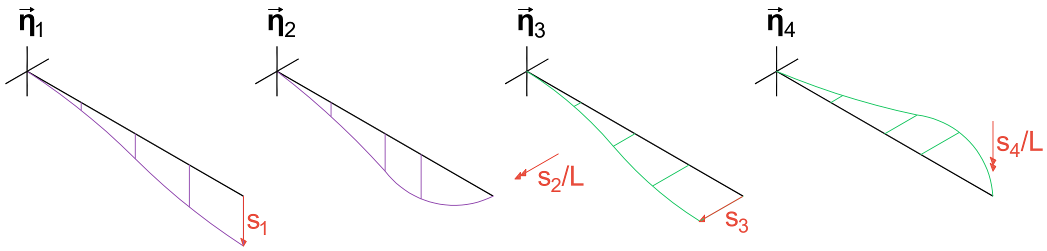


Fig. 3-8 Degree of Freedoms the Cantilever [5]

Within these degrees of freedom, the beam experiences deformations along its length and its boundaries. The corresponding kinematic relationship, of the form $\vec{\boldsymbol{s}}_{(e)} = \boldsymbol{H}_{(e)} \cdot \vec{\boldsymbol{s}}$ from (1.15), is:

$$\begin{bmatrix} w_E \\ \varphi_E L \\ v_E \\ \psi_E L \end{bmatrix}_{(1)} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} s_1 \\ s_2 \\ s_3 \\ s_4 \end{bmatrix} \tag{3.41}$$

The constitutive relationship originates out of (3.36) and takes the form $\vec{\boldsymbol{b}}_{(e)} = \boldsymbol{K}_{(e)} \cdot \vec{\boldsymbol{s}}_{(e)}$ that stems from (1.16).

$$\begin{bmatrix} V_{zE} \\ M_{yE}/L \\ V_{yE} \\ M_{zE}/L \end{bmatrix}_{(1)} = \frac{EI_y}{L^3} \begin{bmatrix} 12 & 6 & -12\omega & 6\omega \\ 6 & 4 & -6\omega & 4\omega \\ -12\omega & -6\omega & 12\alpha & -6\alpha \\ 6\omega & 4\omega & -6\alpha & 4\alpha \end{bmatrix} \begin{bmatrix} w_E \\ \varphi_E L \\ v_E \\ \psi_E L \end{bmatrix}_{(1)} \tag{3.42}$$

The equilibrium condition of the form $\vec{\boldsymbol{b}} = \sum \boldsymbol{H}_{(e)}^T \cdot \vec{\boldsymbol{b}}_{(e)}$ from (1.17) follows to:

$$\begin{bmatrix}1\\0\\0\\0\end{bmatrix} F = \begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}\begin{bmatrix}V_{zE}\\M_{yE}/L\\V_{yE}\\M_{zE}/L\end{bmatrix}_{(1)} \tag{3.43}$$

The system equation of the form $\vec{\boldsymbol{b}} = \boldsymbol{K} \cdot \vec{\boldsymbol{s}}$ results on the basis of (1.18) by inserting (3.41) and (3.42) in (3.43).

$$\begin{bmatrix}1\\0\\0\\0\end{bmatrix} F = \frac{EI_y}{L^3}\begin{bmatrix}12&6&-12\omega&6\omega\\6&4&-6\omega&4\omega\\-12\omega&-6\omega&12\alpha&-6\alpha\\6\omega&4\omega&-6\alpha&4\alpha\end{bmatrix}\begin{bmatrix}s_1\\s_2\\s_3\\s_4\end{bmatrix} \tag{3.44}$$

Solving (3.44) for the unknown displacements yields:

$$\begin{bmatrix}s_1\\s_2\\s_3\\s_4\end{bmatrix} = \frac{1}{\alpha-\omega^2}\begin{bmatrix}\alpha/3\\-\alpha/2\\\omega/3\\\omega/2\end{bmatrix}\cdot\frac{FL^3}{EI_y} \tag{3.45}$$

Whilst the displacements s_1 and s_2 represent bending about the y-axis, s_3 and s_4 represent bending about the z-axis. It is again noticeable, that for $\omega = 0$ unsymmetrical bending, that is the additional bending about the z-axis, would not occur.

Using (3.39), (3.41) and (3.45), the deformations as a function of ξ follow to:

$$\begin{bmatrix}w_z(\xi)\\w_y(\xi)\end{bmatrix} = \frac{3\xi^2-\xi^3}{6(\alpha-\omega^2)}\cdot\begin{bmatrix}\alpha\\\omega\end{bmatrix}\cdot\frac{FL^3}{EI_y} \tag{3.46}$$

Similarly, using (3.40), (3.41) and (3.45), the bending moments as a function of ξ follow to:

$$\begin{bmatrix}M_y(\xi)\\M_z(\xi)\end{bmatrix} = \begin{bmatrix}\xi-1\\0\end{bmatrix}\cdot FL \tag{3.47}$$

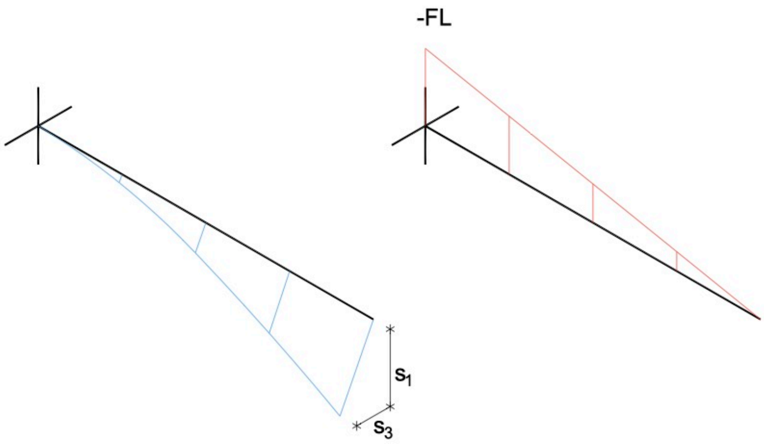


Fig. 3-9 Deformations of the Cantilever [5]

The cantilever can be modeled in the software RSTAB. The chosen section is an EN 10056 L125x75x10 [7] made of S235 steel. The cantilever is given a length of $1m$ and is loaded by a force of $5kN$.The self-weight of the beam is neglected. The parameters to be used are:

$$\alpha = \frac{82.1cm^4}{302cm^4} = 0.2719 \tag{3.48}$$

$$\omega = \frac{89.78cm^4}{302cm^4} = 0.2973 \tag{3.49}$$

$$E = 21000\frac{kN}{cm^2} \tag{3.50}$$

$$L = 100cm \tag{3.51}$$

$$F = 5kN \tag{3.52}$$

The deflections of the free end based on (3.45) are thus:

$$\begin{bmatrix} s_1 \\ s_3 \end{bmatrix} = \begin{bmatrix} 3.89 \\ 4.26 \end{bmatrix} mm \tag{3.53}$$

The deflections given by the software are:

$$\begin{bmatrix} s_1 \\ s_3 \end{bmatrix}_{RSTAB} = \begin{bmatrix} 3.9 \\ 4.3 \end{bmatrix} mm \tag{3.54}$$

The results of the software match the results based on the hand-calculation. The following illustrations of the software analysis correspond to Fig. 3-9.

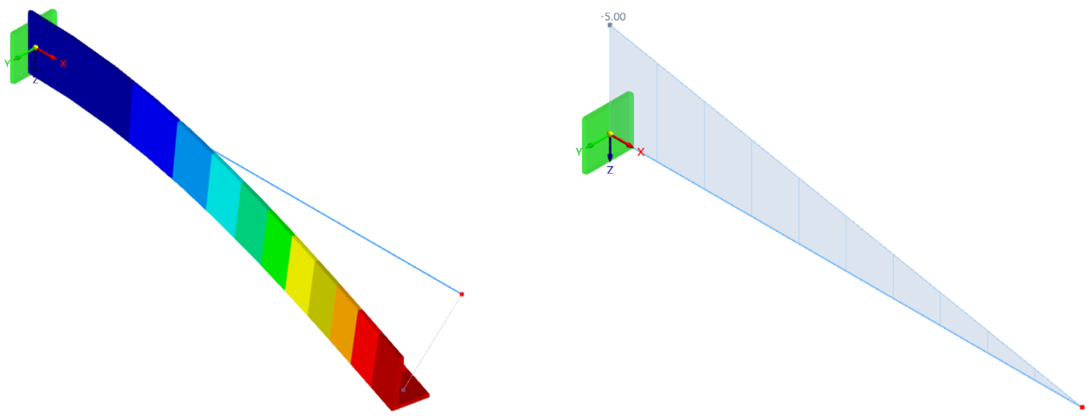


Fig. 3-10 Deformations of the Cantilever in RSTAB [5]

3.3.2 Continuous Purlin

The following scenario represents a purlin resting on an inclined rafter. The purlin spans continuously on three rafters, making it a two-span beam. The distance between each rafter is $4m$. The steel is S235, and the section is an IPE 140 [7]. The beam is subjected to a uniformly distributed load of $4kN/m$.

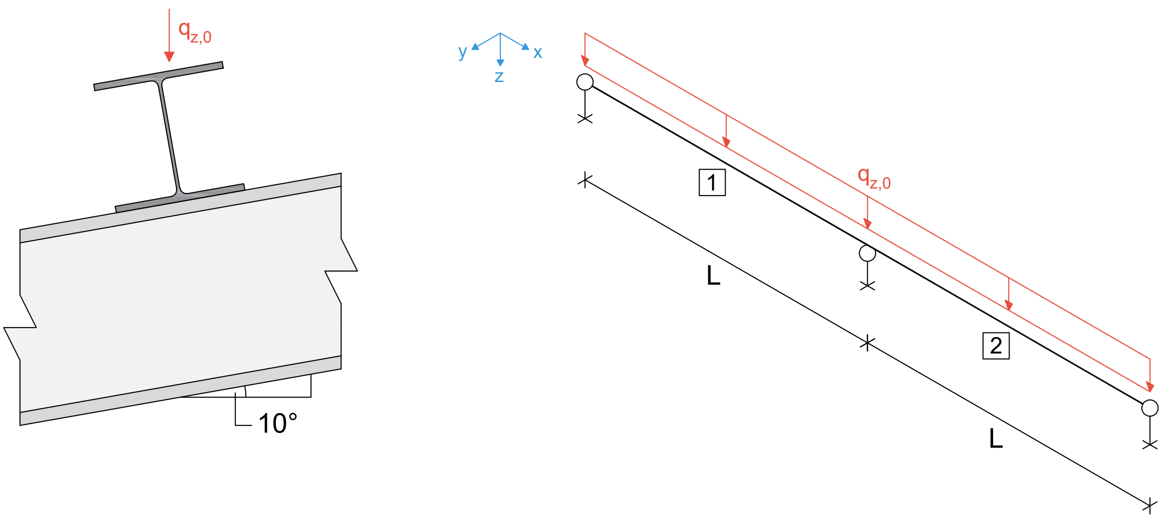


Fig. 3-11 Two-Span Purlin resting on Inclined Rafters [5]

To analyze the beam in its non-principal axes, the section properties must be transformed. This can be achieved analytically using [1, p. 92] or via software using [8]. The non-principal section properties given by [8] are:

$$\begin{bmatrix} I_y \\ I_z \\ I_{yz} \end{bmatrix} = \begin{bmatrix} 526.35 \\ 59.88 \\ 84.89 \end{bmatrix} cm^4 \tag{3.55}$$

The parameters then follow to:

$$\alpha = \frac{59.88cm^4}{526.35cm^4} = 0.1138 \tag{3.56}$$

$$\omega = \frac{84.89cm^4}{526.35cm^4} = 0.1613 \tag{3.57}$$

$$E = 21000\frac{kN}{cm^2} \tag{3.58}$$

$$L = 400cm \tag{3.59}$$

$$q_{z,0} = 4\frac{kN}{m} \tag{3.60}$$

The system has six degrees of freedom. However, symmetric and asymmetric degrees of freedom are chosen. Since the system and its external load are symmetric, only the symmetric degrees of freedom are activated. Thus, the asymmetric degrees of freedom are neglected. The remaining two symmetric degrees of freedom are illustrated below.

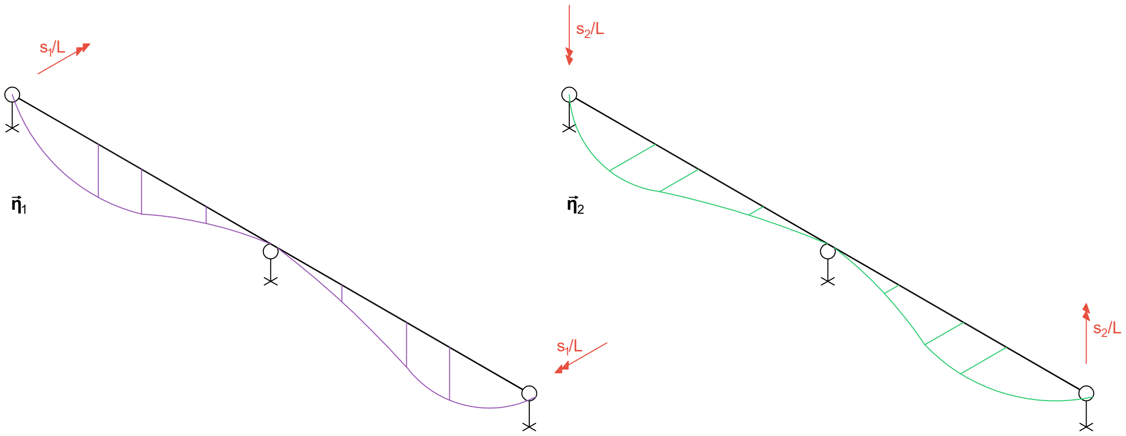


Fig. 3-12 Symmetric Degrees of Freedom [5]

The kinematic relationships of the form $\vec{\boldsymbol{s}}_{(e)} = \boldsymbol{H}_{(e)} \cdot \vec{\boldsymbol{s}}$ are:

$$\begin{bmatrix} \varphi_A L \\ \psi_A L \end{bmatrix}_{(1)} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} s_1 \\ s_2 \end{bmatrix} \tag{3.61}$$

$$\begin{bmatrix} \varphi_E L \\ \psi_E L \end{bmatrix}_{(2)} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} s_1 \\ s_2 \end{bmatrix} \tag{3.62}$$

The constitutive relationships originate out of (3.36) and (3.38). They take the form $\vec{\boldsymbol{b}} = \boldsymbol{K} \cdot \vec{\boldsymbol{s}} + \vec{\boldsymbol{b}}_0$ as described in (1.12).

$$\begin{bmatrix} M_{yA}/L \\ M_{zA}/L \end{bmatrix}_{(1)} = \frac{EI_y}{L^3} \begin{bmatrix} 4 & 4\omega \\ 4\omega & 4\alpha \end{bmatrix} \begin{bmatrix} \varphi_A L \\ \psi_A L \end{bmatrix}_{(1)} + \frac{EI_y}{L^2} \begin{bmatrix} 1/12 \\ 0 \end{bmatrix} \rho_{z,0} \tag{3.63}$$

$$\begin{bmatrix} M_{yE}/L \\ M_{zE}/L \end{bmatrix}_{(2)} = \frac{EI_y}{L^3}\begin{bmatrix} 4 & 4\omega \\ 4\omega & 4\alpha \end{bmatrix}\begin{bmatrix} \varphi_E L \\ \psi_E L \end{bmatrix}_{(2)} + \frac{EI_y}{L^2}\begin{bmatrix} -1/12 \\ 0 \end{bmatrix}\rho_{z,0} \tag{3.64}$$

Based on (1.18), the system equation is formed by (3.61) to (3.64). Since there are no isolated loads, the force vector is solely based on the vectors $\vec{\boldsymbol{b}}_{\mathbf{0}(e)}$.

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \frac{EI_y}{L^3}\begin{bmatrix} 8 & -8\omega \\ -8\omega & 8\alpha \end{bmatrix}\begin{bmatrix} s_1 \\ s_2 \end{bmatrix} - \frac{EI_y}{L^2}\begin{bmatrix} 1/6 \\ 0 \end{bmatrix}\rho_{z,0} \tag{3.65}$$

Solving (3.65) for the unknown displacements gives:

$$\begin{bmatrix} s_1 \\ s_2 \end{bmatrix} = \frac{1}{48(\alpha-\omega^2)}\begin{bmatrix} \alpha \\ \omega \end{bmatrix}\frac{q_{z,0}L^4}{EI_y} \tag{3.66}$$

Using (3.39), (3.61) to (3.64), the deformations as a function of ξ follow to:

$$\begin{bmatrix} w_z(\xi) \\ w_y(\xi) \end{bmatrix}_{(1)} = \frac{2\xi^4 - 3\xi^3 + \xi}{48(\alpha-\omega^2)} \cdot \begin{bmatrix} \alpha \\ \omega \end{bmatrix} \cdot \frac{q_{z,0}L^4}{EI_y} \tag{3.67}$$

$$\begin{bmatrix} w_z(\xi) \\ w_y(\xi) \end{bmatrix}_{(2)} = \frac{2\xi^4 - 5\xi^3 + 3\xi^2}{48(\alpha-\omega^2)} \cdot \begin{bmatrix} \alpha \\ \omega \end{bmatrix} \cdot \frac{q_{z,0}L^4}{EI_y} \tag{3.68}$$

Similarly, using (3.40), (3.61) to (3.64), the bending moments as a function of ξ follow to:

$$\begin{bmatrix} M_y(\xi) \\ M_z(\xi) \end{bmatrix}_{(1)} = \frac{3\xi - 4\xi^2}{8} \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} \cdot q_{z,0}L^2 \tag{3.69}$$

$$\begin{bmatrix} M_y(\xi) \\ M_z(\xi) \end{bmatrix}_{(2)} = \frac{5\xi - 4\xi^2 - 1}{8} \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} \cdot q_{z,0}L^2 \tag{3.70}$$

Plugging in the values of the parameters into the solutions above leads to the following results.

$$\begin{bmatrix} w_{z,max} \\ w_{y,max} \\ M_{y,max,I} \\ M_{y,max,II} \end{bmatrix} = \begin{bmatrix} 6.5mm \\ 9.2mm \\ 4.5kNm \\ -8kNm \end{bmatrix} \tag{3.71}$$

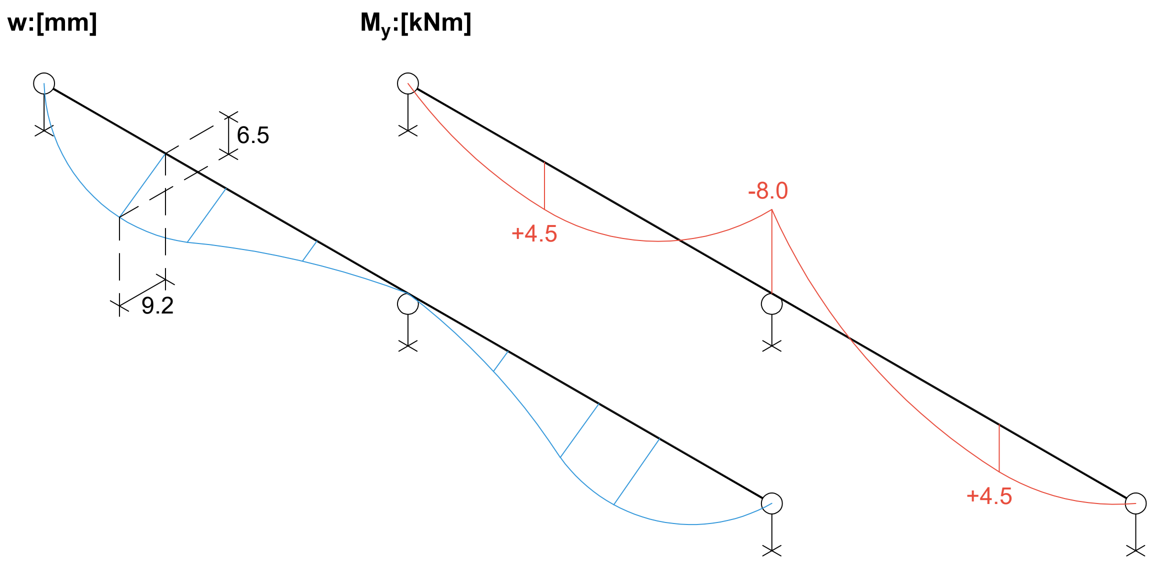


Fig. 3-13 Results of the Hand-Calculation [5]

The results of the analysis using RSTAB are given below. The results are accurate. The images produced by the software correspond to those of Fig. 3-13.

$$\begin{bmatrix} w_{z,max} \\ w_{y,max} \\ M_{y,max,I} \\ M_{y,max,II} \end{bmatrix}_{RSTAB} = \begin{bmatrix} 6.5mm \\ 9.2mm \\ 4.48kNm \\ -8kNm \end{bmatrix} \tag{3.72}$$

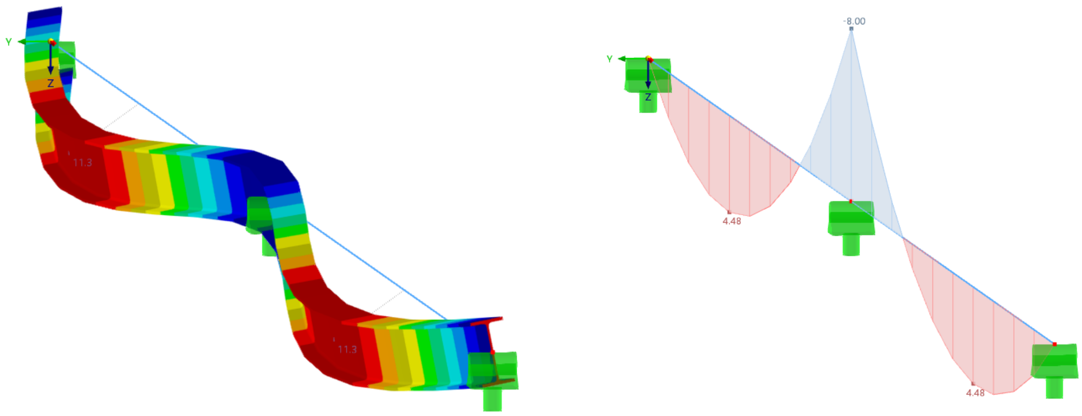


Fig. 3-14 Results of the Software Analysis [5]

4 Second Order Analysis

4.1 Transition from First to Second Order Theory

4.1.1 Condition of a Beam

Within this section, the beam is subjected to an axial compressive force P. An infinitesimal beam element and its internal forces are illustrated below. The compressive force acts through the centroid of the cross-section.

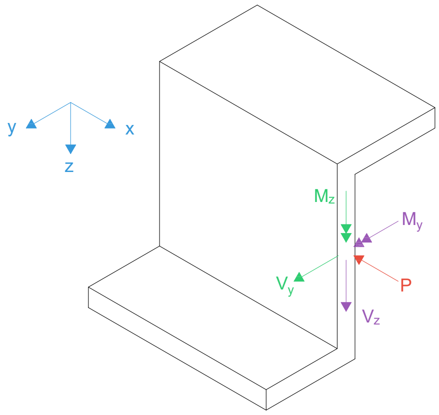


Fig. 4-1 Beam Element with Compressive Force [5]

With

P ... Axial Compressive Force

4.1.2 Substitute Forces

As described in 2.1, there exist three order of theories that provide the description of deformations of a system. Second Order Theory dictates that when a column laterally deforms, the column head moves laterally and axially. In comparison, an axial movement is not considered in First Order Theory. The deformations are illustrated below.

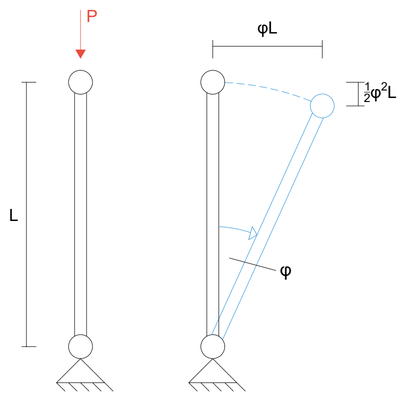


Fig. 4-2 Second Order Theory – Deformation of a Column [5]

The axial force experiences a deformation parallel to itself. Thus, a potential is created by the axial force. This potential, denoted by Π_P, is the negative product of the force and the deformation parallel to the force.

$$\Pi_P = -\frac{1}{2} P \varphi^2 L \tag{4.1}$$

With

Π_P ... Potential of an axial load because of Second Order Theory

The additional potential is due to the kinematics described by Second Order Theory. However, instead of describing the effect of Second Order Theory through a potential, substitute forces, Q, can be applied that mimic the same effect. These forces, that act as a couple, allow the system to be analyzed based on First Order Theory.

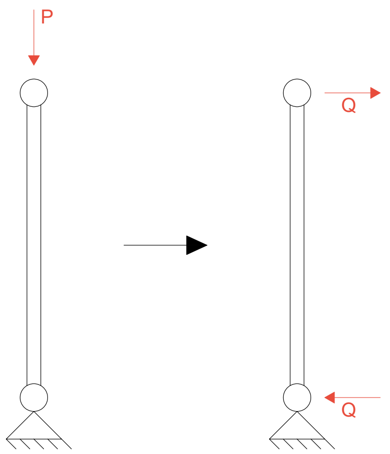


Fig. 4-3 Substitute Forces of a Column [5]

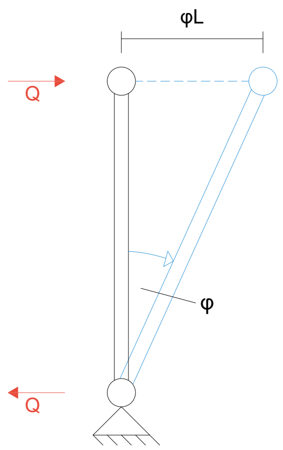


Fig. 4-4 First Order Theory Deformation of a Column with Substitute Forces [5]

With

Q ... Substitute force that mirrors the effect of Second Order Theory

Where

$$\mathrm{Q} := P \cdot \varphi \tag{4.2}$$

The substitute force is dependent on the axial load P and the rotation of the column φ. Since it is dependent on a deformation, it creates an active potential when experiencing the lateral sway of the column. The potential of the substitute forces is:

$$\Pi_Q := -\int P \, ds = -\int P \, \varphi \, d\varphi \cdot L = -\frac{1}{2} P \varphi^2 L \tag{4.3}$$

This is the same potential shown in (4.1). The effect of Second Order Theory can therefore be reproduced by introducing substitute forces.

4.1.3 Substitute Moments

Within this section, Second Order Theory is applied to a beam. Specifically, the effect of Second Order Theory is applied to the individual fibers in correspondence with the Beam-Fiber Model described in 2.4. When a beam is subjected to an axial compressive force, this force is equally distributed on the fibers. With n amount of fibers, an arbitrary fiber receives the axial force ΔP.

$$\Delta P := \frac{P}{n} \tag{4.4}$$

Since the Beam-Fiber Model allows the fibers of the beam to sway freely, they act like pinned columns subject to compression, as described in 4.1.2. Henceforth, they exhibit substitute forces that mimic the effect of Second Order Theory. Each fiber has substitute forces, Q_z and Q_y, respectively acting in z and y direction. By use of (4.2) and Fig. 4-5 the substitute forces of an arbitrary fiber are:

$$\begin{bmatrix} \Delta Q_z \\ \Delta Q_y \end{bmatrix} = \Delta P \cdot \frac{\partial}{\partial x} \begin{bmatrix} w_z \\ w_y \end{bmatrix} \tag{4.5}$$

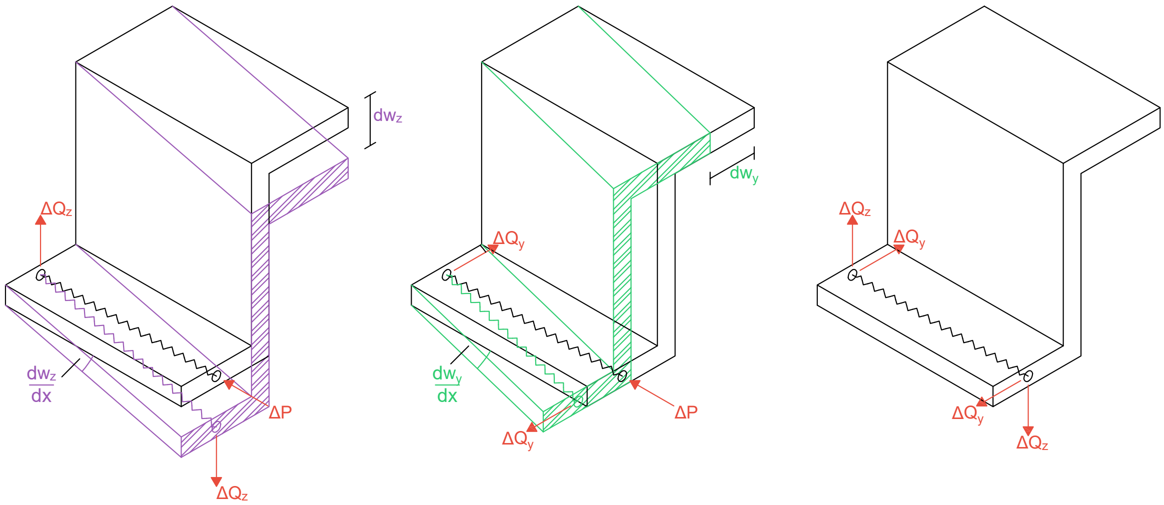


Fig. 4-5 Sways of the Cross-Section and Substitute Forces of an arbitrary Fiber [5]

The summation of all substitute forces of the individual fibers results in the substitute forces of the entire cross-section. Moreover, the couples of substitute forces create a moment about each axis, as illustrated below. This demonstrates that due to Second Order Theory, an axial loaded beam is subjected to additional bending moments at each infinitesimal section. This is the main effect of Second Order Theory that subsequently leads to a stability problem known as Buckling ([1], [2]). The substitute moments used for further analysis are then described by the following.

$$\begin{bmatrix} Q_z \\ Q_y \end{bmatrix} = \begin{bmatrix} \Delta Q_z \\ \Delta Q_y \end{bmatrix} = \mathrm{n} \cdot \frac{P}{n} \cdot \frac{\partial}{\partial x} \begin{bmatrix} w_z \\ w_y \end{bmatrix} = P \cdot \frac{\partial}{\partial x} \begin{bmatrix} w_z \\ w_y \end{bmatrix} \tag{4.6}$$

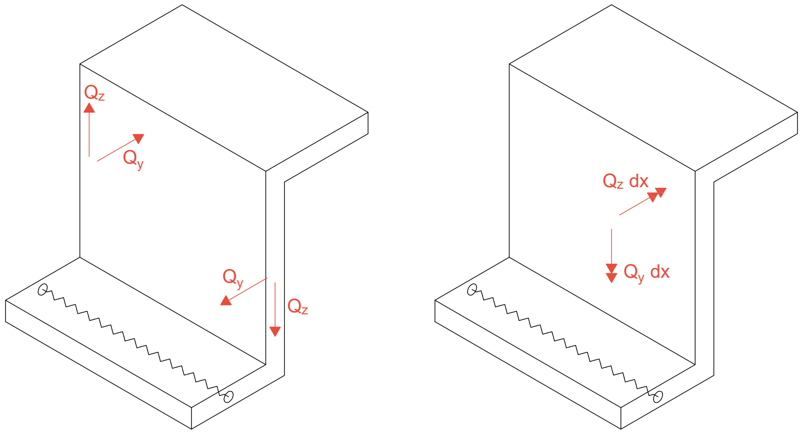


Fig. 4-6 Substitute Forces (left) and Substitute Moments (right) [5]

4.2 System of Differential Equations

4.2.1 Derivation

A beam with the following internal and external forces is considered. As derived in 4.1.3, the beam is subjected to additional moments originating out of an axial force, which is assumed to act constant along the beam's length.

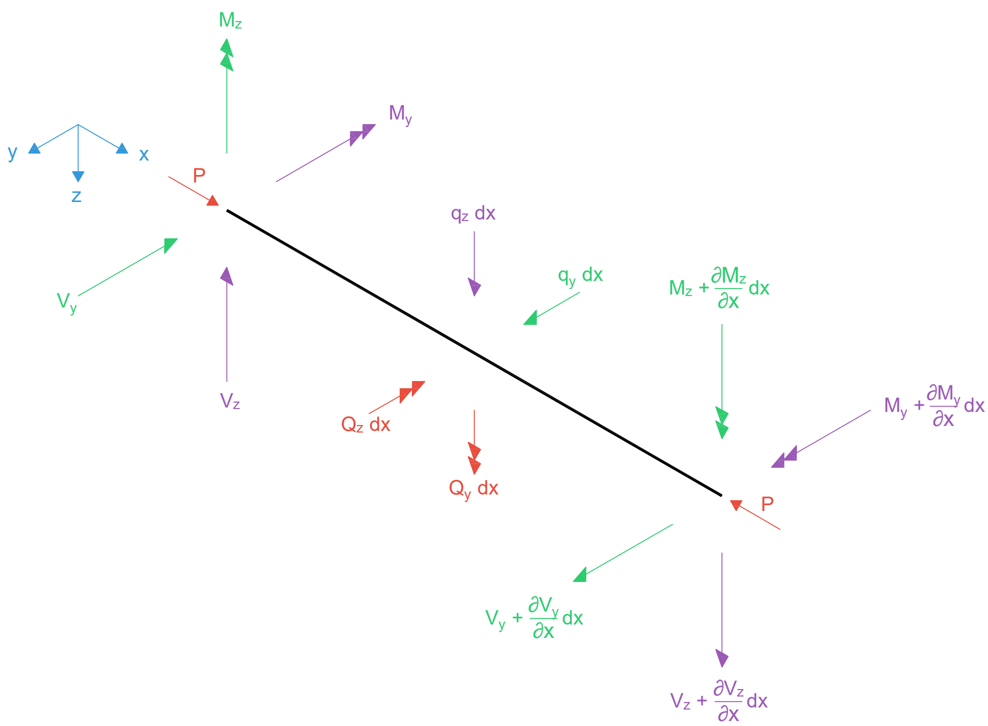


Fig. 4-7 Infinitesimal Beam Element with Internal and External Forces [5]

The equilibrium of forces in z and y direction yields:

$$\frac{\partial}{\partial x}\begin{bmatrix} V_z \\ V_y \end{bmatrix} = -\begin{bmatrix} q_z \\ q_y \end{bmatrix} \tag{4.7}$$

The equilibrium of moments about the y- and z-axes gives:

$$\frac{\partial}{\partial x}\begin{bmatrix} M_y \\ M_z \end{bmatrix} + \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} Q_z \\ Q_y \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} V_z \\ V_y \end{bmatrix} \tag{4.8}$$

The substitute forces are given in (4.6), which gives the following when combined with (4.8).

$$\frac{\partial}{\partial x}\begin{bmatrix} M_y \\ M_z \end{bmatrix} + P \cdot \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\frac{\partial}{\partial x}\begin{bmatrix} w_z \\ w_y \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} V_z \\ V_y \end{bmatrix} \tag{4.9}$$

Differentiating (4.9) with respect to x allows the insertion of (4.7).

$$\frac{\partial^2}{\partial x^2}\begin{bmatrix} M_y \\ M_z \end{bmatrix} + P \cdot \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\frac{\partial^2}{\partial x^2}\begin{bmatrix} w_z \\ w_y \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} q_z \\ q_y \end{bmatrix} \tag{4.10}$$

The second derivatives of the moments were derived in (3.4), whilst the curvatures are given in (3.5). Together, they can be inserted in (4.10), which gives:

$$E \cdot \begin{bmatrix} -I_y & I_{yz} \\ -I_{yz} & I_z \end{bmatrix}\frac{\partial^4}{\partial x^4}\begin{bmatrix} w_z \\ w_y \end{bmatrix} + P \cdot \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\frac{\partial^2}{\partial x^2}\begin{bmatrix} w_z \\ w_y \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} q_z \\ q_y \end{bmatrix} \tag{4.11}$$

This is equivalent to:

$$\frac{\partial^4}{\partial x^4}\begin{bmatrix} w_z \\ w_y \end{bmatrix} + \frac{1}{E(I_y I_z - {I_{yz}}^2)}\begin{bmatrix} I_z & I_{yz} \\ I_{yz} & I_y \end{bmatrix}\left[P \cdot \frac{\partial^2}{\partial x^2}\begin{bmatrix} w_z \\ w_y \end{bmatrix} - \begin{bmatrix} q_z \\ q_y \end{bmatrix}\right] = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{4.12}$$

The above system of differential equations describes an axially compressed beam subject to unsymmetrical bending. It takes the general form:

$$\frac{\partial^4}{\partial x^4}\vec{\boldsymbol{w}} + \frac{\boldsymbol{J}'}{E\Delta} \cdot \left[P \cdot \frac{\partial^2}{\partial x^2}\vec{\boldsymbol{w}} - \vec{\boldsymbol{q}}\right] = \vec{\boldsymbol{0}} \tag{4.13}$$

With

Δ ... Determinant of the inertia tensor

Where

$$\Delta := I_y I_z - {I_{yz}}^2 \tag{4.14}$$

$$\boldsymbol{J}' := \begin{bmatrix} I_z & I_{yz} \\ I_{yz} & I_y \end{bmatrix} \tag{4.15}$$

Using the parameters from (3.8) and (3.9), as well as the definition from (3.10) and (3.11), the system of differential equations can be parameterized. Additionally, a new parameter, the beam's stability identification value, ε, is defined as:

$$\varepsilon := \sqrt{\frac{PL^2}{EI_y}} \tag{4.16}$$

The system of differential equations then follows to:

$$\frac{\partial^4}{\partial\xi^4}\begin{bmatrix} w_z \\ w_y \end{bmatrix} + \frac{1}{\alpha-\omega^2}\cdot\begin{bmatrix} \alpha & \omega \\ \omega & 1 \end{bmatrix}\cdot\left[\varepsilon^2\cdot\frac{\partial^2}{\partial\xi^2}\begin{bmatrix} w_z \\ w_y \end{bmatrix} - \frac{L^4}{EI_y}\cdot\begin{bmatrix} q_z \\ q_y \end{bmatrix}\right] = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{4.17}$$

4.2.2 Solution

Within the scope of this thesis, only the homogeneous solution will be focused upon. The homogeneous system of differential equations consists of two ordinary homogeneous linear fourth order differential equations with constant coefficients.

$$\frac{\partial^4}{\partial\xi^4}\vec{\boldsymbol{w}} + \frac{1}{\alpha-\omega^2}\cdot\begin{bmatrix} \alpha & \omega \\ \omega & 1 \end{bmatrix}\cdot\varepsilon^2\cdot\frac{\partial^2}{\partial\xi^2}\vec{\boldsymbol{w}} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{4.18}$$

The solution is less trivial than that of First Order Theory and its solution is a fundamental topic of this thesis. The above system of differential equations resembles the type found in [9]. Thus, a similar Ansatz is chosen to find the eight distinct vectors that satisfy the system. The Ansatz is hereby chosen as:

$$\vec{\boldsymbol{w}}\cdot\frac{1}{L} = \vec{\boldsymbol{\phi}}\cdot e^{\theta\xi} \tag{4.19}$$

Inserting (4.19) in (4.18) gives:

$$\theta^4\cdot\vec{\boldsymbol{\phi}}\cdot e^{\theta\xi} + \frac{\varepsilon^2}{\alpha-\omega^2}\cdot\begin{bmatrix} \alpha & \omega \\ \omega & 1 \end{bmatrix}\cdot\theta^2\cdot\vec{\boldsymbol{\phi}}\cdot e^{\theta\xi} = \vec{\boldsymbol{0}} \tag{4.20}$$

Since

$$e^{\theta\xi} \neq 0 \ \forall\ \theta \in \mathbb{C} \tag{4.21}$$

it follows that

$$\theta^2\cdot\left[\theta^2\cdot\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \frac{\varepsilon^2}{\alpha-\omega^2}\cdot\begin{bmatrix} \alpha & \omega \\ \omega & 1 \end{bmatrix}\right]\cdot\vec{\boldsymbol{\phi}} = \vec{\boldsymbol{0}} \tag{4.22}$$

For (4.22) to be true, $\vec{\boldsymbol{\phi}} = \vec{\boldsymbol{0}}$ resembles the trivial solution that leads nowhere. The solution $\theta^2 = 0$ corresponds to the solution of four vectors, $\vec{\boldsymbol{w}}_k$, that fulfill the system of differential equations. Due to $\theta = 0$, these vectors are non-exponential functions, but rather polynomials:

$$\vec{\boldsymbol{w}}_k = \left\{\begin{bmatrix} \xi \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ \xi \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right\} \tag{4.23}$$

The remaining four vectors arise out of the condition below.

$$\left[\theta^2 \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \frac{\varepsilon^2}{\alpha - \omega^2} \cdot \begin{bmatrix} \alpha & \omega \\ \omega & 1 \end{bmatrix}\right] \cdot \vec{\boldsymbol{\phi}} = \vec{\mathbf{0}} \tag{4.24}$$

This mimics an Eigenproblem. To prevent the trivial solution of $\vec{\boldsymbol{\phi}} = \vec{\mathbf{0}}$, the determinant of the matrix left of the vector $\vec{\boldsymbol{\phi}}$ must be equal to zero. Then and only then does no inverse matrix exist. The following condition must be met.

$$det\left[\theta^2 \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \frac{\varepsilon^2}{\alpha - \omega^2} \cdot \begin{bmatrix} \alpha & \omega \\ \omega & 1 \end{bmatrix}\right] = 0 \tag{4.25}$$

This leads to the characteristic polynomial:

$$\theta^4 + \theta^2 \cdot \varepsilon^2 \cdot \frac{\alpha + 1}{\alpha - \omega^2} + \frac{\varepsilon^4}{\alpha - \omega^2} = 0 \tag{4.26}$$

The Eigenvalues, i.e. the solutions of (4.26), follow to:

$$\theta = \pm\, \varepsilon \cdot i \cdot \sqrt{\frac{1}{2} \cdot \frac{\alpha + 1}{\alpha - \omega^2} \mp \sqrt{\frac{1}{4} \cdot \left(\frac{\alpha + 1}{\alpha - \omega^2}\right)^2 - \frac{1}{\alpha - \omega^2}}} \tag{4.27}$$

Where

$$i := \sqrt{-1} \tag{4.28}$$

These are subsequently defined by the parameters β_1 and β_2.

$$\theta = \pm\, \varepsilon \cdot i \cdot \beta_{1,2} \tag{4.29}$$

Where

$$\beta_{1,2} := \sqrt{\frac{1}{2} \cdot \frac{\alpha + 1}{\alpha - \omega^2} \mp \sqrt{\frac{1}{4} \cdot \left(\frac{\alpha + 1}{\alpha - \omega^2}\right)^2 - \frac{1}{\alpha - \omega^2}}} \tag{4.30}$$

The Eigenvectors that satisfy (4.24) are:

$$\vec{\boldsymbol{\phi}} = \left\{\begin{bmatrix} \gamma_1 \\ 1 \end{bmatrix}, \begin{bmatrix} \gamma_2 \\ 1 \end{bmatrix}\right\} \tag{4.31}$$

Where

$$\gamma_{1,2} := \frac{2\omega}{1 - \alpha \pm \sqrt{\alpha^2 - 2\alpha + \omega^2 + 1}} \tag{4.32}$$

The remaining four vectors that satisfy the system of differential equations consist of one eigenvector from (4.31) and one eigenvalue from (4.27). They take the form of the Ansatz from (4.19). The complex solutions may be rewritten as real solutions in the form of trigonometric functions, as done in [10]. The real set of solutions follows to:

$$\vec{\boldsymbol{w}}_k = \left\{\begin{bmatrix}\gamma_1\\1\end{bmatrix}\cdot\cos(\beta_1\varepsilon\xi)\,,\begin{bmatrix}\gamma_1\\1\end{bmatrix}\cdot\sin(\beta_1\varepsilon\xi)\,,\begin{bmatrix}\gamma_2\\1\end{bmatrix}\cdot\cos(\beta_2\varepsilon\xi)\,,\begin{bmatrix}\gamma_2\\1\end{bmatrix}\cdot\sin(\beta_2\varepsilon\xi)\right\} \tag{4.33}$$

The vectors $\vec{\boldsymbol{w}}_k$ from (4.23) and (4.33) make up the homogeneous solution. Through superposition, the homogeneous solution takes the form:

$$\vec{\boldsymbol{w}}_{hom} = \sum_{k=1}^{8} \vec{\boldsymbol{w}}_k \cdot c_k \tag{4.34}$$

and results in:

$$\begin{aligned}\vec{\boldsymbol{w}}_{hom}\cdot\frac{1}{L} = \begin{bmatrix}\gamma_1\\1\end{bmatrix}\cdot\cos(\beta_1\varepsilon\xi)\cdot c_1 + \begin{bmatrix}\gamma_1\\1\end{bmatrix}\cdot\sin(\beta_1\varepsilon\xi)\cdot c_2 + \begin{bmatrix}\gamma_2\\1\end{bmatrix}\cdot\cos(\beta_2\varepsilon\xi)\cdot c_3 \\ + \begin{bmatrix}\gamma_2\\1\end{bmatrix}\sin(\beta_2\varepsilon\xi)\cdot c_4 + \begin{bmatrix}\xi\\0\end{bmatrix}\cdot c_5 + \begin{bmatrix}1\\0\end{bmatrix}\cdot c_6 + \begin{bmatrix}0\\\xi\end{bmatrix}\cdot c_7 + \begin{bmatrix}0\\1\end{bmatrix}\cdot c_8\end{aligned} \tag{4.35}$$

4.2.3 Deformations

The solution presented in (4.35) may be rewritten to give the deformations dependent on the constants of integration. For simplicity, the trigonometric terms are replaced by the placeholders $\bar{c}_1(\xi)$, $\bar{c}_2(\xi)$, $\bar{s}_1(\xi)$ and $\bar{s}_2(\xi)$.

$$\begin{bmatrix}w_z\\w_y\end{bmatrix}\cdot\frac{1}{L} = \begin{bmatrix}\gamma_1\bar{c}_1(\xi) & \gamma_1\bar{s}_1(\xi) & \gamma_2\bar{c}_2(\xi) & \gamma_2\bar{s}_2(\xi) & \xi & 1 & 0 & 0\\ \bar{c}_1(\xi) & \bar{s}_1(\xi) & \bar{c}_2(\xi) & \bar{s}_2(\xi) & 0 & 0 & \xi & 1\end{bmatrix}\begin{bmatrix}c_1\\c_2\\c_3\\c_4\\c_5\\c_6\\c_7\\c_8\end{bmatrix} \tag{4.36}$$

Where

$$\bar{c}_{1,2}(\xi) := \cos\left(\beta_{1,2}\varepsilon\xi\right) \tag{4.37}$$

$$\bar{s}_{1,2}(\xi) := \sin\left(\beta_{1,2}\varepsilon\xi\right) \tag{4.38}$$

The tilts are obtained as the first derivatives with respect to x. In accordance with (3.11), the derivative with respect to x is appropriately changed to with respect to ξ.

$$\begin{bmatrix}w_z'L\\w_y'L\end{bmatrix}\frac{1}{L} = \begin{bmatrix}-\gamma_1\bar{s}_1(\xi)\beta_1\varepsilon & \gamma_1\bar{c}_1(\xi)\beta_1\varepsilon & -\gamma_2\bar{s}_2(\xi)\beta_2\varepsilon & \gamma_2\bar{c}_2(\xi)\beta_2\varepsilon & 1 & 0 & 0 & 0\\ -\bar{s}_1(\xi)\beta_1\varepsilon & \bar{c}_1(\xi)\beta_1\varepsilon & -\bar{s}_2(\xi)\beta_2\varepsilon & \bar{c}_2(\xi)\beta_2\varepsilon & 0 & 0 & 1 & 0\end{bmatrix}\begin{bmatrix}c_1\\c_2\\c_3\\c_4\\c_5\\c_6\\c_7\\c_8\end{bmatrix} \tag{4.39}$$

Letting ξ be equal to zero and one, gives the relation between the boundary displacement variables and the integration constants. It takes the form $\vec{\boldsymbol{u}} = \boldsymbol{U} \cdot \vec{\boldsymbol{c}} + \vec{\boldsymbol{u}}_{inhom}$. For simplicity, the trigonometric terms are replaced by the placeholders $\bar{c}_1$, $\bar{c}_2$, $\bar{s}_1$ and $\bar{s}_2$.

$$\begin{bmatrix} w_z(0) \\ w_z'(0)L \\ w_z(1) \\ w_z'(1)L \\ w_y(0) \\ w_y'(0)L \\ w_y(1) \\ w_y'(1)L \end{bmatrix} \cdot \frac{1}{L} = \begin{bmatrix} \gamma_1 & 0 & \gamma_2 & 0 & 0 & 1 & 0 & 0 \\ 0 & \gamma_1\beta_1\varepsilon & 0 & \gamma_2\beta_2\varepsilon & 1 & 0 & 0 & 0 \\ \gamma_1\bar{c}_1 & \gamma_1\bar{s}_1 & \gamma_2\bar{c}_2 & \gamma_2\bar{s}_2 & 1 & 1 & 0 & 0 \\ -\gamma_1\bar{s}_1\beta_1\varepsilon & \gamma_1\bar{c}_1\beta_1\varepsilon & -\gamma_2\bar{s}_2\beta_2\varepsilon & \gamma_2\bar{c}_2\beta_2\varepsilon & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & \beta_1\varepsilon & 0 & \beta_2\varepsilon & 0 & 0 & 1 & 0 \\ \bar{c}_1 & \bar{s}_1 & \bar{c}_2 & \bar{s}_2 & 0 & 0 & 1 & 1 \\ -\bar{s}_1\beta_1\varepsilon & \bar{c}_1\beta_1\varepsilon & -\bar{s}_2\beta_2\varepsilon & \bar{c}_2\beta_2\varepsilon & 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \\ c_6 \\ c_7 \\ c_8 \end{bmatrix} \tag{4.40}$$

4.2.4 Internal Forces

To determine the internal moments of the beam, the second derivative of the deformations are needed. These follow to:

$$\begin{bmatrix} w_z'' \\ w_y'' \end{bmatrix} L = \begin{bmatrix} -\gamma_1\bar{c}_1(\xi)\beta_1^2\varepsilon^2 & -\gamma_1\bar{s}_1(\xi)\beta_1^2\varepsilon^2 & -\gamma_2\bar{c}_2(\xi)\beta_2^2\varepsilon^2 & -\gamma_2\bar{s}_2(\xi)\beta_2^2\varepsilon^2 \\ -\bar{c}_1(\xi)\beta_1^2\varepsilon^2 & -\bar{s}_1(\xi)\beta_1^2\varepsilon^2 & -\bar{c}_2(\xi)\beta_2^2\varepsilon^2 & -\bar{s}_2(\xi)\beta_2^2\varepsilon^2 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} \tag{4.41}$$

The parameterized Moment-Curvature Relationship stems from (3.26).

$$\begin{bmatrix} M_y \\ M_z \end{bmatrix} = \frac{EI_y}{L} \cdot \begin{bmatrix} -1 & \omega \\ -\omega & \alpha \end{bmatrix} \begin{bmatrix} w_z'' \\ w_y'' \end{bmatrix} L \tag{4.42}$$

Combining (4.41) and (4.42) gives the moments dependent on the constants of integration.

$$\begin{bmatrix} M_y \\ M_z \end{bmatrix} = \frac{EI_y}{L} \begin{bmatrix} (\gamma_1-\omega)\bar{c}_1(\xi)\beta_1^2\varepsilon^2 & (\gamma_1-\omega)\bar{s}_1(\xi)\beta_1^2\varepsilon^2 & (\gamma_2-\omega)\bar{c}_2(\xi)\beta_2^2\varepsilon^2 & (\gamma_2-\omega)\bar{s}_2(\xi)\beta_2^2\varepsilon^2 \\ (\gamma_1\omega-\alpha)\bar{c}_1(\xi)\beta_1^2\varepsilon^2 & (\gamma_1\omega-\alpha)\bar{c}_1(\xi)\beta_1^2\varepsilon^2 & (\gamma_2\omega-\alpha)\bar{c}_2(\xi)\beta_2^2\varepsilon^2 & (\gamma_2\omega-\alpha)\bar{s}_2(\xi)\beta_2^2\varepsilon^2 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} \tag{4.43}$$

The shear forces are determined through the expression from (3.29). They follow to:

$$\begin{bmatrix} V_z \\ V_y \end{bmatrix} = \frac{EI_y}{L^2} \begin{bmatrix} (\omega-\gamma_1)\bar{s}_1(\xi)\beta_1^3\varepsilon^3 & (\gamma_1-\omega)\bar{c}_1(\xi)\beta_1^3\varepsilon^3 & (\omega-\gamma_2)\bar{s}_2(\xi)\beta_2^3\varepsilon^3 & (\gamma_2-\omega)\bar{c}_2(\xi)\beta_2^3\varepsilon^3 \\ (\gamma_1\omega-\alpha)\bar{s}_1(\xi)\beta_1^3\varepsilon^3 & (\alpha-\gamma_1\omega)\bar{c}_1(\xi)\beta_1^3\varepsilon^3 & (\gamma_2\omega-\alpha)\bar{s}_2(\xi)\beta_2^3\varepsilon^3 & (\alpha-\gamma_2\omega)\bar{c}_2(\xi)\beta_2^3\varepsilon^3 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} \tag{4.44}$$

Letting ξ be equal to zero and one gives the relation between the internal boundary force variables and the constants of integration.

$$\begin{bmatrix} V_z(0) \\ M_y(0)/L \\ V_z(1) \\ M_y(1)/L \\ V_y(0) \\ M_z(0)/L \\ V_y(1) \\ M_z(1)/L \end{bmatrix} = \frac{EI_y}{L^2} \begin{bmatrix} 0 & (\gamma_1-\omega)\beta_1^3\varepsilon^3 & 0 & (\gamma_2-\omega)\beta_2^3\varepsilon^3 \\ (\gamma_1-\omega)\beta_1^2\varepsilon^2 & 0 & (\gamma_2-\omega)\beta_2^2\varepsilon^2 & 0 \\ (\omega-\gamma_1)\bar{s}_1\beta_1^3\varepsilon^3 & (\gamma_1-\omega)\bar{c}_1\beta_1^3\varepsilon^3 & (\omega-\gamma_2)\bar{s}_2\beta_2^3\varepsilon^3 & (\gamma_2-\omega)\bar{c}_2\beta_2^3\varepsilon^3 \\ (\gamma_1-\omega)\bar{c}_1\beta_1^2\varepsilon^2 & (\gamma_1-\omega)\bar{s}_1\beta_1^2\varepsilon^2 & (\gamma_2-\omega)\bar{c}_2\beta_2^2\varepsilon^2 & (\gamma_2-\omega)\bar{s}_2\beta_2^2\varepsilon^2 \\ 0 & (\alpha-\gamma_1\omega)\beta_1^3\varepsilon^3 & 0 & (\alpha-\gamma_2\omega)\beta_2^3\varepsilon^3 \\ (\gamma_1\omega-\alpha)\beta_1^2\varepsilon^2 & 0 & (\gamma_2\omega-\alpha)\beta_2^2\varepsilon^2 & 0 \\ (\gamma_1\omega-\alpha)\bar{s}_1\beta_1^3\varepsilon^3 & (\alpha-\gamma_1\omega)\bar{c}_1\beta_1^3\varepsilon^3 & (\gamma_2\omega-\alpha)\bar{s}_2\beta_2^3\varepsilon^3 & (\alpha-\gamma_2\omega)\bar{c}_2\beta_2^3\varepsilon^3 \\ (\gamma_1\omega-\alpha)\bar{c}_1\beta_1^2\varepsilon^2 & (\gamma_1\omega-\alpha)\bar{s}_1\beta_1^2\varepsilon^2 & (\gamma_2\omega-\alpha)\bar{c}_2\beta_2^2\varepsilon^2 & (\gamma_2\omega-\alpha)\bar{s}_2\beta_2^2\varepsilon^2 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} \tag{4.45}$$

The substitute forces are additional components that are necessary to determine the constitutive relationship. The substitute forces are given by (4.6). Through rearrangement, the substitute forces are proportional to ε^2.

$$\begin{bmatrix} Q_z \\ Q_y \end{bmatrix} = P \cdot \frac{\partial}{\partial x}\begin{bmatrix} w_z \\ w_y \end{bmatrix} = \frac{EI_y}{L^2}\cdot\frac{PL^2}{EI_y}\frac{\partial}{\partial x}\begin{bmatrix} w_z \\ w_y \end{bmatrix} = \frac{EI_y}{L^2}\cdot\varepsilon^2\cdot\begin{bmatrix} w_z'L \\ w_y'L \end{bmatrix}\frac{1}{L} \tag{4.46}$$

By use of (4.39), the substitute forces evaluated at zero and one are:

$$\begin{bmatrix} Q_z(0) \\ Q_z(1) \\ Q_y(0) \\ Q_y(1) \end{bmatrix} = \frac{EI_y}{L^2} \begin{bmatrix} 0 & \gamma_1\beta_1\varepsilon^3 & 0 & \gamma_2\beta_2\varepsilon^3 & \varepsilon^2 & 0 & 0 & 0 \\ -\gamma_1\bar{s}_1\beta_1\varepsilon^3 & \gamma_1\bar{c}_1\beta_1\varepsilon^3 & -\gamma_2\bar{s}_2\beta_2\varepsilon^3 & \gamma_2\bar{c}_2\beta_2\varepsilon^3 & \varepsilon^2 & 0 & 0 & 0 \\ 0 & \beta_1\varepsilon^3 & 0 & \beta_2\varepsilon^3 & 0 & 0 & \varepsilon^2 & 0 \\ -\bar{s}_1\beta_1\varepsilon^3 & \bar{c}_1\beta_1\varepsilon^3 & -\bar{s}_2\beta_2\varepsilon^3 & \bar{c}_2\beta_2\varepsilon^3 & 0 & 0 & \varepsilon^2 & 0 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \\ c_6 \\ c_7 \\ c_8 \end{bmatrix} \tag{4.47}$$

4.3 Displacement Method

4.3.1 Definition of Displacements

The same definitions of displacements presented in 3.2.1, will be used. The relation between displacements and boundary deformations takes the form $\vec{\boldsymbol{s}} = \boldsymbol{S}\cdot\vec{\boldsymbol{u}}$.

$$\begin{bmatrix} w_A \\ \varphi_A L \\ w_E \\ \varphi_E L \\ v_A \\ \psi_A L \\ v_E \\ \psi_E L \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} w_z(0) \\ w_z'(0)L \\ w_z(1) \\ w_z'(1)L \\ w_y(0) \\ w_y'(0)L \\ w_y(1) \\ w_y'(1)L \end{bmatrix} \tag{4.48}$$

4.3.2 Definition of Forces

A similar definition of forces, as presented in 3.2.2, will be used. Though, the substitute forces must be considered. As shown in Fig. 4-8, when the beam is displaced as to cause a constant tilt of the beam, the axial force is displaced as well. The equilibrium of the deformed system yields additional support reactions. These support reactions are the manifestation of the substitute forces.

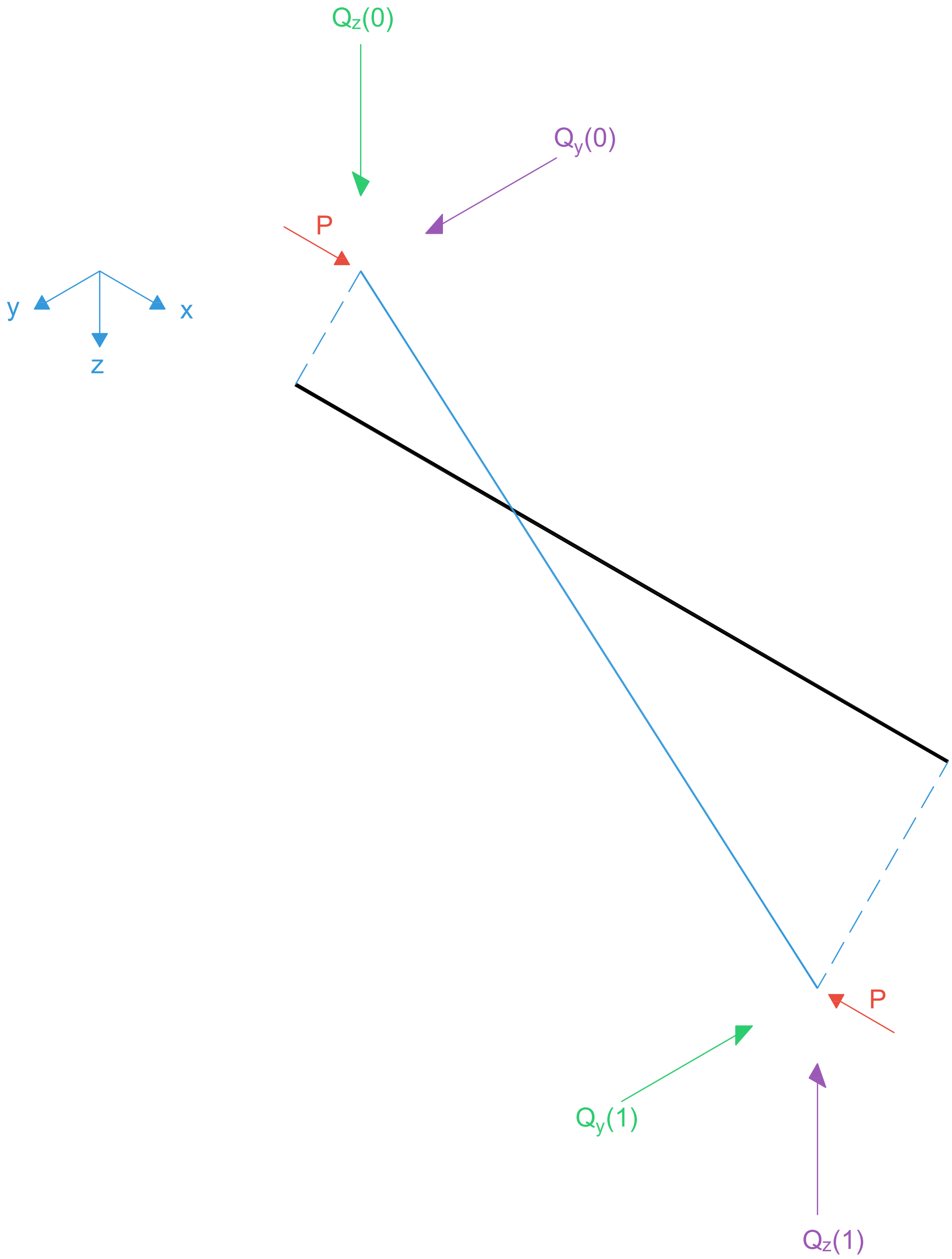


Fig. 4-8 Additional Support Reactions through Substitute Forces [5]

When comparing Fig. 3-3 with Fig. 4-8, the substitute forces at the supports act antiparallel to the shear forces of the beam. The relation of the form $\vec{\boldsymbol{b}} = \boldsymbol{B} \cdot \vec{\boldsymbol{r}}$ is therefore:

$$\begin{bmatrix} V_{zA} \\ M_{yA}/L \\ V_{zE} \\ M_{yE}/L \\ V_{yA} \\ M_{zA}/L \\ V_{yE} \\ M_{zE}/L \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} V_z(0) - Q_z(0) \\ M_y(0)/L \\ V_z(1) - Q_z(1) \\ M_y(1)/L \\ V_y(0) - Q_y(0) \\ M_z(0)/L \\ V_y(1) - Q_y(1) \\ M_z(1)/L \end{bmatrix} \tag{4.49}$$

4.3.3 Constitutive Relationship

The stiffness matrix within this section is of Second Order Theory and shall be denoted by $\boldsymbol{K^{II}}$. The stiffness matrix is determined by (1.13) with the matrices $\boldsymbol{B}$, $\boldsymbol{R}$, $\boldsymbol{U}$ and $\boldsymbol{S}$ originating from (4.49), (4.45) and (4.47), (4.40), and (4.48), respectively. The constitutive relationship of Second Order Theory takes the form $\vec{\boldsymbol{b}}^{\boldsymbol{II}} = \boldsymbol{K^{II}} \cdot \vec{\boldsymbol{s}}^{\boldsymbol{II}}$ and becomes:

$$\begin{bmatrix} V_{zA} \\ M_{yA}/L \\ V_{zE} \\ M_{yE}/L \\ V_{yA} \\ M_{zA}/L \\ V_{yE} \\ M_{zE}/L \end{bmatrix} = \frac{EI_y}{L^3} \cdot \begin{bmatrix} \boldsymbol{K_y^{II}} & \boldsymbol{K_{yz}^{II}} \\ \left(\boldsymbol{K_{yz}^{II}}\right)^T & \boldsymbol{K_z^{II}} \end{bmatrix} \cdot \begin{bmatrix} w_A \\ \varphi_A L \\ w_E \\ \varphi_E L \\ v_A \\ \psi_A L \\ v_E \\ \psi_E L \end{bmatrix} \tag{4.50}$$

The submatrices $\boldsymbol{K_y^{II}}$, $\boldsymbol{K_z^{II}}$ and $\boldsymbol{K_{yz}^{II}}$ consist of stiffness coefficients κ and λ. These are a function of the beam stability identification value, ε, and the parameters α and ω. The submatrices can be expressed as:

$$\boldsymbol{K_y^{II}} := \begin{bmatrix} \kappa_{y,V} & -\kappa_{y,K} & -\kappa_{y,V} & -\kappa_{y,K} \\ -\kappa_{y,K} & \kappa_{y,M} & \kappa_{y,K} & \kappa_{y,M}^* \\ -\kappa_{y,V} & \kappa_{y,K} & \kappa_{y,V} & \kappa_{y,K} \\ -\kappa_{y,K} & \kappa_{y,M}^* & \kappa_{y,K} & \kappa_{y,M} \end{bmatrix} \tag{4.51}$$

$$\boldsymbol{K_z^{II}} := \begin{bmatrix} \kappa_{z,V} & \kappa_{z,K} & -\kappa_{z,V} & \kappa_{z,K} \\ \kappa_{z,K} & \kappa_{z,M} & -\kappa_{z,K} & \kappa_{z,M}^* \\ -\kappa_{z,V} & -\kappa_{z,K} & \kappa_{z,V} & -\kappa_{z,K} \\ \kappa_{z,K} & \kappa_{z,M}^* & -\kappa_{z,K} & \kappa_{z,M} \end{bmatrix} \tag{4.52}$$

$$\boldsymbol{K}_{yz}^{II} := \begin{bmatrix} -\lambda_V & -\lambda_K & \lambda_V & -\lambda_K \\ \lambda_K & \lambda_M & -\lambda_K & \lambda_M^* \\ \lambda_V & \lambda_K & -\lambda_V & \lambda_K \\ \lambda_K & \lambda_M^* & -\lambda_K & \lambda_M \end{bmatrix} \tag{4.53}$$

With

κ ... Stiffness coefficients for $\boldsymbol{K}_y^{II}$ and $\boldsymbol{K}_z^{II}$

λ ... Stiffness coefficients for $\boldsymbol{K}_{yz}^{II}$

The definitions of these stiffness coefficients are given in Attachment C. The following figures showcase the typical graphs of the stiffness coefficients.

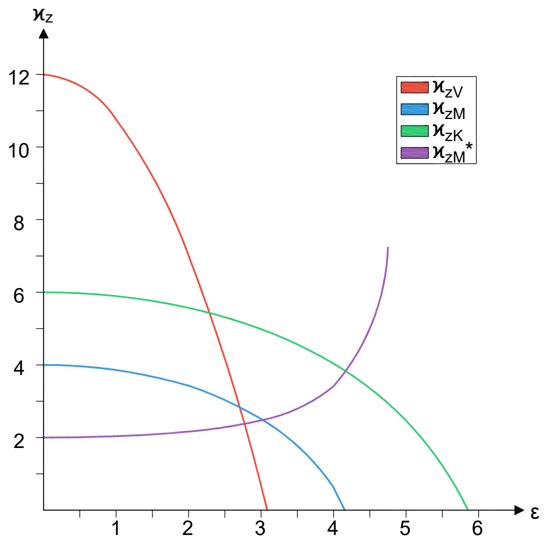


Fig. 4-9 Stiffness Coefficients for $\boldsymbol{K}_y^{II}$ [5]

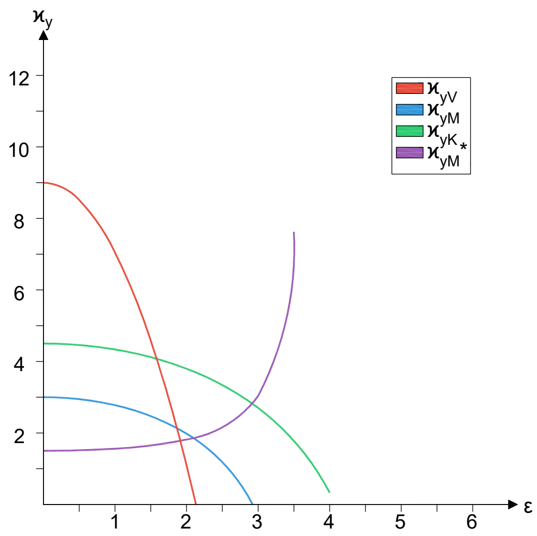


Fig. 4-10 Stiffness Coefficients for $\boldsymbol{K}_z^{II}$ [5]

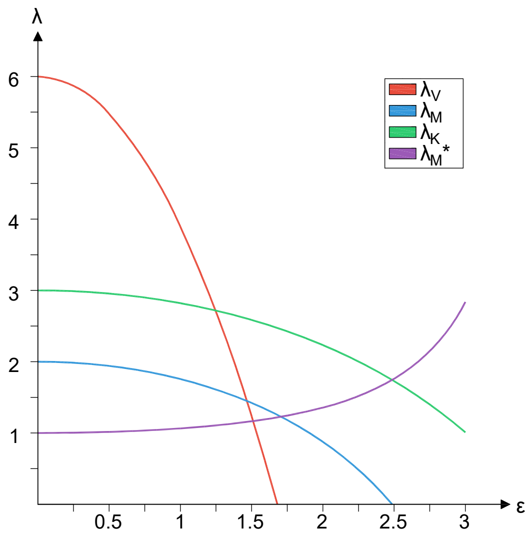


Fig. 4-11 Stiffness Coefficients for $\boldsymbol{K}_{yz}^{II}$ [5]

The stiffness coefficients are dependent on ε, α and ω. For $\omega = 0$, the stiffness coefficients λ go to zero. This property demonstrates their plausibility, since for $\omega = 0$, they must be equal to zero. In general, an increase in α leads to a horizontal stretching of the function. Dissimilarly, an increase in ω leads to a horizontal compression of the graph. This is demonstrated in Fig. 4-12 through Fig. 4-14 for select values of α and ω on the stiffness coefficients $\kappa_{y,M}$, $\kappa_{z,M}$ and λ_M.

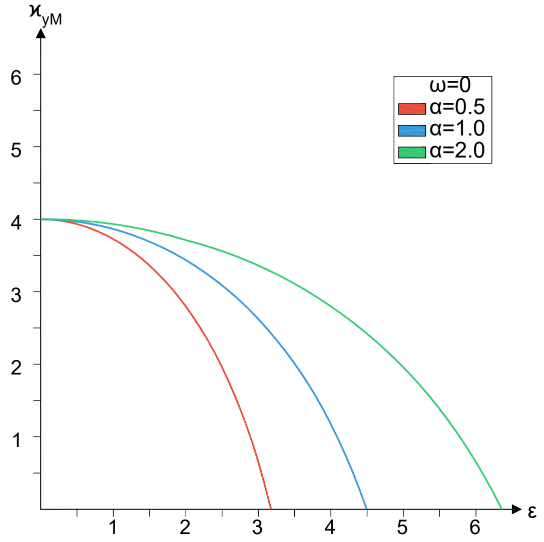


Fig. 4-12 $\kappa_{y,M}$ for select values [5]

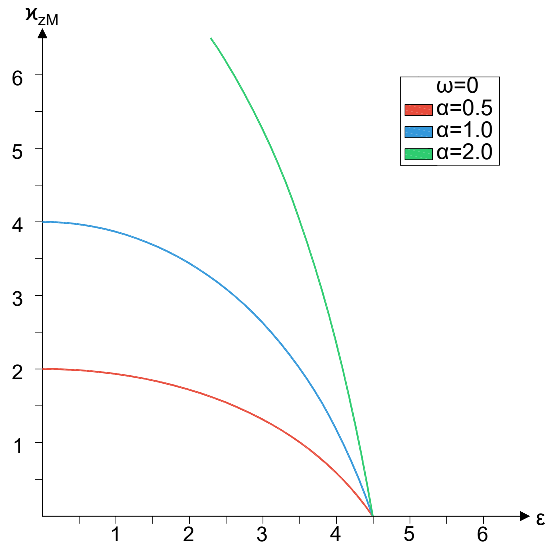


Fig. 4-13 $\kappa_{z,M}$ for select values [5]

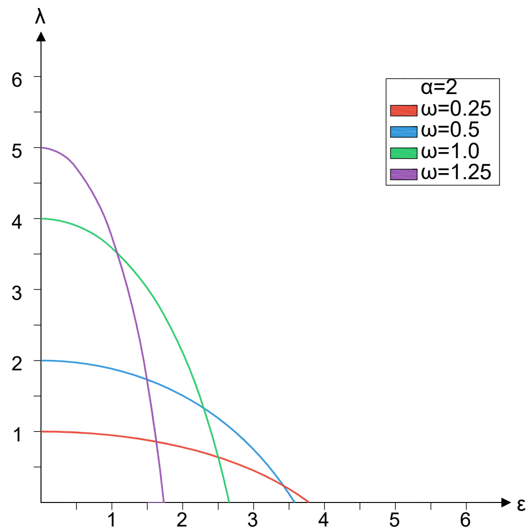


Fig. 4-14 λ_M for select values [5]

When a beam is not subject to an axial compressive force, the beam's stability identification number, ε, goes to zero. For this special case, the stiffness matrix of Second Order Theory converges onto the stiffness matrix of First Order Theory. This is formally expressed by the following limit. Graphically it can be seen in Fig. 4-9 through Fig. 4-11, that for $\varepsilon = 0$, the values taken by the stiffness coefficients are precisely those of First Order Theory.

$$\lim_{\varepsilon \to 0} \boldsymbol{K^{II}}(\varepsilon) = \boldsymbol{K^{I}} \tag{4.54}$$

4.3.4 Internal Forces

Within this section, the internal bending moments of the beam are given a relation with the boundary displacements. Using (4.43), (4.40) and (4.48), the bending moments can be expressed through:

$$\begin{bmatrix} M_y \\ M_z \end{bmatrix} = \frac{EI_y}{L^2} \begin{bmatrix} \mu^{II}_{w_A} & \mu^{II}_{\varphi_A} & \mu^{II}_{w_E} & \mu^{II}_{\varphi_E} & \mu^{II}_{wv_A} & \mu^{II}_{\varphi\psi_A} & \mu^{II}_{wv_E} & \mu^{II}_{\varphi\psi_E} \\ -\mu^{II}_{wv_A} & \mu^{II}_{\varphi\psi_A} & -\mu^{II}_{wv_E} & \mu^{II}_{\varphi\psi_E} & \mu^{II}_{v_A} & \mu^{II}_{\psi_A} & \mu^{II}_{v_E} & \mu^{II}_{\psi_E} \end{bmatrix} \begin{bmatrix} w_A \\ \varphi_A L \\ w_E \\ \varphi_E L \\ v_A \\ \psi_A L \\ v_E \\ \psi_E L \end{bmatrix} \tag{4.55}$$

The amplitude functions are of Second Order Theory and are denoted as such. Like the stiffness parameters, they are expressed through functional coefficients and are given in Attachment D. Illustrated below, are the graphs of $\mu^{II}_{w_A}$ for different values of ε. For $\varepsilon = 0$, the graph approaches the amplitude function of First Order Theory, $\mu^{I}_{w_A}$.

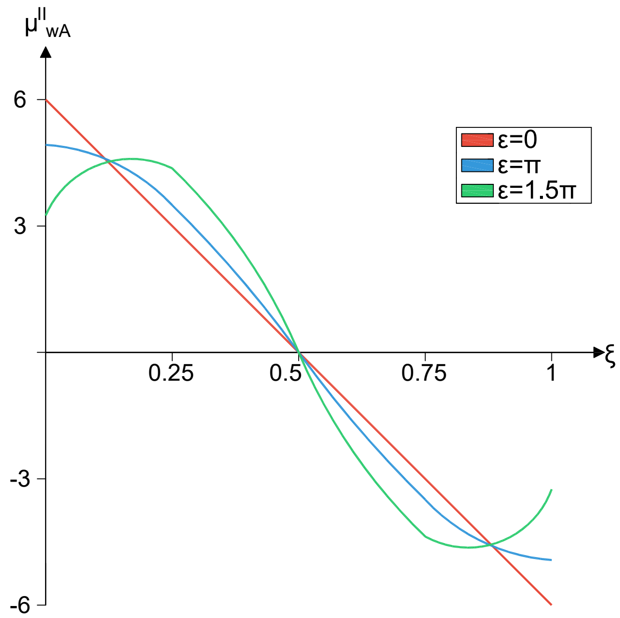


Fig. 4-15 $\mu^{II}_{w_A}$ for select values [5]

4.4 Exemplary Analyses

4.4.1 Pinned-Pinned Column

The following system illustrates a simply supported beam-column that is subjected to an axial load P.

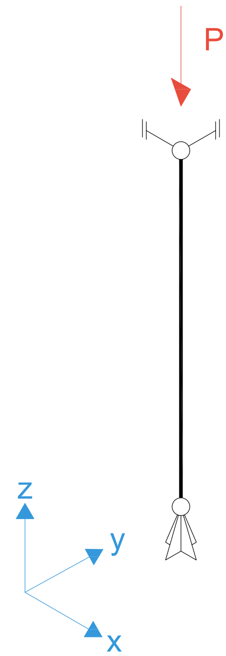


Fig. 4-16 Axially Loaded Simply Supported Beam-Column [5]

This system has four distinct degrees of freedom. These can be chosen to be symmetric and antisymmetric, as shown below.

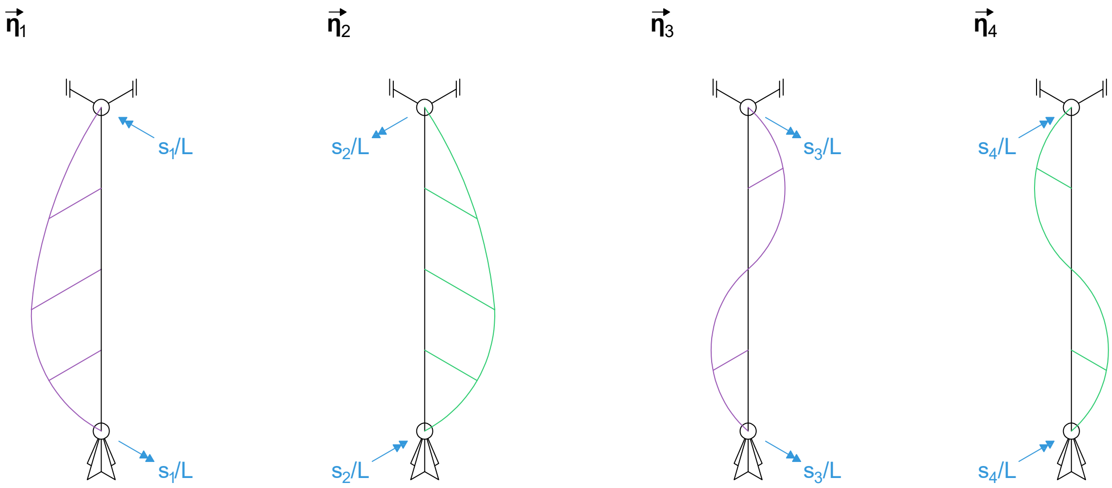


Fig. 4-17 Symmetric and Antisymmetric Degrees of Freedom [5]

The kinematic relationship of the form $\vec{\boldsymbol{s}}_{(e)} = \boldsymbol{H}_{(e)} \cdot \vec{\boldsymbol{s}}$ from (1.15), is:

$$\begin{bmatrix} \varphi_A L \\ \varphi_E L \\ \psi_A L \\ \psi_E L \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} s_1 \\ s_2 \\ s_3 \\ s_4 \end{bmatrix} \tag{4.56}$$

The constitutive relationship originates out of (4.50) and takes the form $\vec{\boldsymbol{b}}_{(e)} = \boldsymbol{K}_{(e)} \cdot \vec{\boldsymbol{s}}_{(e)}$.

$$\begin{bmatrix} M_{yA}/L \\ M_{yE}/L \\ M_{zA}/L \\ M_{zE}/L \end{bmatrix} = \frac{EI_y}{L^3} \begin{bmatrix} \kappa_{y,M} & \kappa_{y,M}^* & \lambda_M & \lambda_M^* \\ \kappa_{y,M}^* & \kappa_{y,M} & \lambda_M^* & \lambda_M \\ \lambda_M & \lambda_M^* & \kappa_{z,M} & \kappa_{z,M}^* \\ \lambda_M^* & \lambda_M & \kappa_{z,M}^* & \kappa_{z,M} \end{bmatrix} \begin{bmatrix} w_E \\ \varphi_E L \\ v_E \\ \psi_E L \end{bmatrix} \tag{4.57}$$

The system's stiffness matrix is based on (1.13) and results to the following, when combining (4.56) and (4.57).

$$\boldsymbol{K} = \frac{EI_y}{L^3} \cdot 2 \cdot \begin{bmatrix} \kappa_{y,M} - \kappa_{y,M}^* & \lambda_M - \lambda_M^* & 0 & 0 \\ \lambda_M - \lambda_M^* & \kappa_{z,M} - \kappa_{z,M}^* & 0 & 0 \\ 0 & 0 & \kappa_{y,M} + \kappa_{y,M}^* & \lambda_M + \lambda_M^* \\ 0 & 0 & \lambda_M + \lambda_M^* & \kappa_{z,M} + \kappa_{z,M}^* \end{bmatrix} \begin{bmatrix} s_1 \\ s_2 \\ s_3 \\ s_4 \end{bmatrix} \tag{4.58}$$

The determinant of the system's stiffness matrix represents the overall stiffness of the system. Here, the determinant is:

$$\begin{aligned} det(\boldsymbol{K}) = 16 \cdot \left[\left(\kappa_{y,M} - \kappa_{y,M}^*\right) \cdot \left(\kappa_{z,M} - \kappa_{z,M}^*\right) - \left(\lambda_M - \lambda_M^*\right)^2 \right] \\ \cdot \left[\left(\kappa_{y,M} + \kappa_{y,M}^*\right) \cdot \left(\kappa_{z,M} + \kappa_{z,M}^*\right) - \left(\lambda_M + \lambda_M^*\right)^2 \right] \end{aligned} \tag{4.59}$$

For select values of α and ω, the determinant is plotted and portrayed in the following figure.

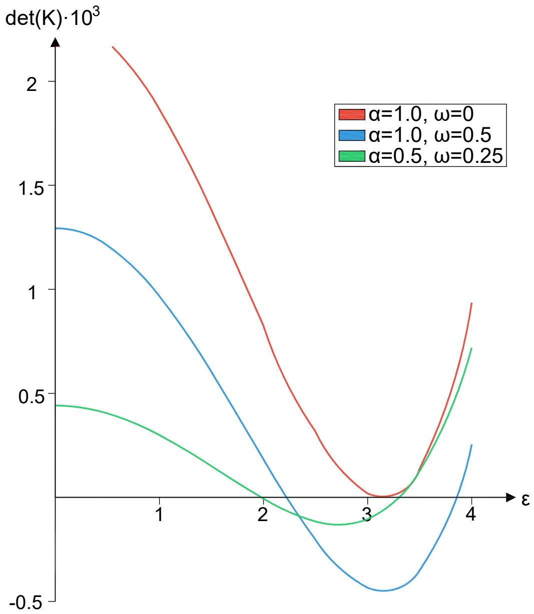


Fig. 4-18 Graph of $det(\boldsymbol{K})$ for select values [5]

An instability in the system occurs when the determinant is equal to zero. The corresponding critical beam stability identification value is denoted by ε_{cr}. Using (4.59), the critical values can be gathered for various α and ω. The resulting values are presented graphically in Fig. 4-19

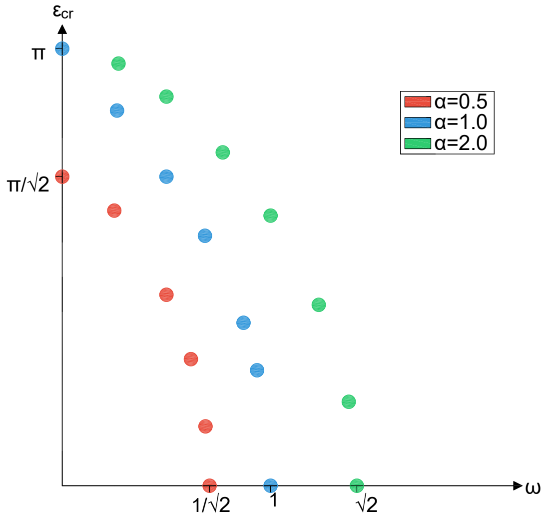


Fig. 4-19 Plot of Gathered ε_{cr} [5]

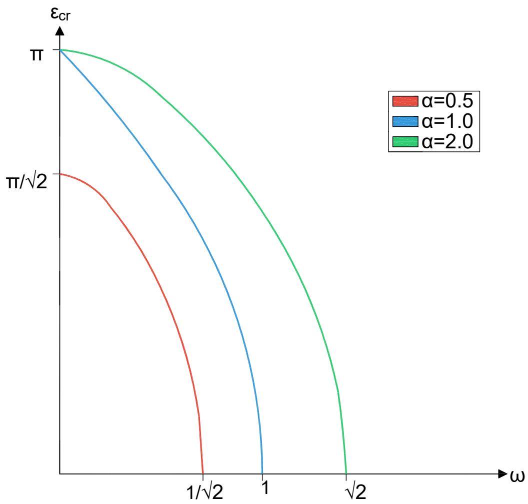


Fig. 4-20 Plot of $\varepsilon_{cr}(\alpha, \omega)$ [5]

It is apparent that the critical values plot an apparently well-defined family of curves. This family of curves can be expressed through the following function.

$$\varepsilon_{cr}(\alpha, \omega) = \varepsilon_{cr}(\alpha, \omega = 0) \cdot \sqrt{\frac{1+\alpha}{2} - \sqrt{\left(\frac{1+\alpha}{2}\right)^2 + \omega^2}} \tag{4.60}$$

This function originates out of the equality, that regardless of what axes the beam is analyzed in, the critical load P_{cr} must remain the same. That is:

$$P_{cr} = [\varepsilon_{cr}(\alpha, \omega)]^2 \cdot \frac{EI_y}{L^2} = const. \tag{4.61}$$

The unique case for $\omega = 0$ represents an analysis made in the beam's principal axes. In such axes, the principal second moments of inertia are used. The critical load is the smallest load needed to cause failure and is correspondingly proportional to the minor principal moment of inertia. Through [1, p. 93], this can be expressed by:

$$I_{min} = \frac{I_y + I_z}{2} - \sqrt{\left(\frac{I_y - I_z}{2}\right)^2 + I_{yz}^2} \tag{4.62}$$

The critical load remains constant. Therefore:

$$P_{cr} = (\varepsilon_{cr}(\alpha, \omega))^2 \cdot \frac{EI_y}{L^2} = (\varepsilon_{cr}(\alpha, \omega = 0))^2 \cdot \frac{EI_{min}}{L^2} \tag{4.63}$$

Plugging (4.62) into (4.63), simplifying and parameterizing yields the function showcased in (4.60). All that is necessary, is the value of ε_{cr} when $\omega = 0$.

On a sidenote, this analysis serves as a proof of the buckling load of the Euler Case II [1, p. 248]. It can be seen in Fig. 4-18, that for $\omega = 0$ and $\alpha \geq 1$, the critical beam stability identification number is π. Through (4.61), the buckling load is then:

$$P_{cr} = \pi^2 \cdot \frac{EI_y}{L^2} \tag{4.64}$$

4.4.2 Fixed Free Column

The following scenario features a column that is fixed at one end and free at the other. The beam-column is subject to an axial force and a transverse load F_z.

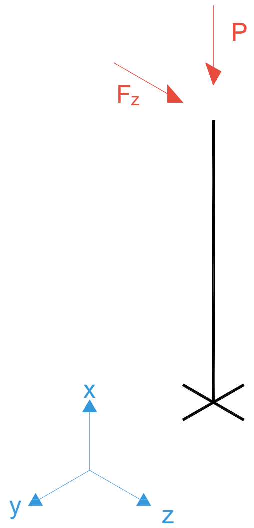


Fig. 4-21 Fixed-Free Beam-Column [5]

The system has four distinct degrees of freedom. These are illustrated below.

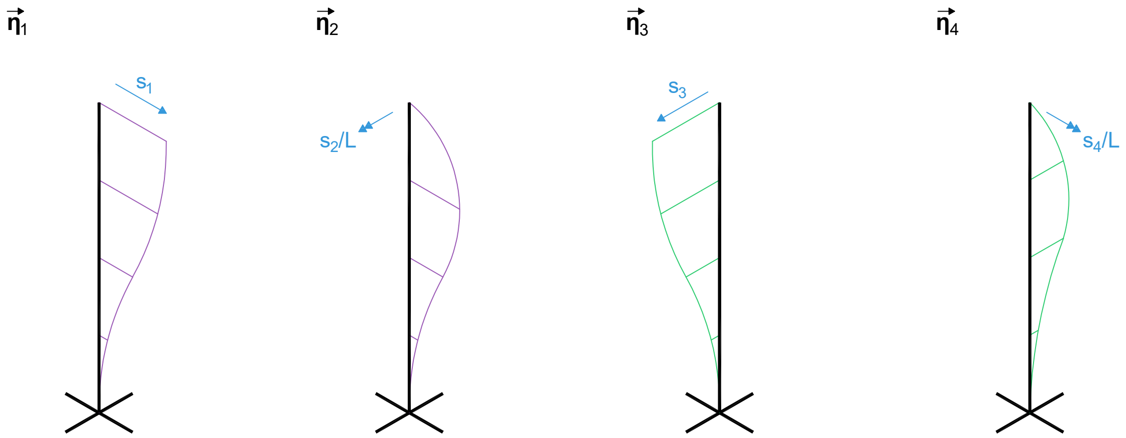


Fig. 4-22 Degrees of Freedom of the Beam-Column [5]

The kinematic relationship of the form $\vec{s}_{(e)} = \boldsymbol{H}_{(e)} \cdot \vec{s}$ from (1.15), is:

$$\begin{bmatrix} w_E \\ \varphi_E L \\ v_E \\ \psi_E L \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} s_1 \\ s_2 \\ s_3 \\ s_4 \end{bmatrix} \tag{4.65}$$

The constitutive relationship originates out of (4.50) and takes the form $\vec{b}_{(e)} = \boldsymbol{K}_{(e)} \cdot \vec{s}_{(e)}$.

$$\begin{bmatrix} V_{zE} \\ M_{yE}/L \\ V_{yE} \\ M_{zE}/L \end{bmatrix} = \frac{EI_y}{L^3} \begin{bmatrix} \kappa_{y,V} & \kappa_{y,K} & -\lambda_V & \lambda_K \\ \kappa_{y,K} & \kappa_{y,M} & -\lambda_K & \lambda_M \\ -\lambda_V & -\lambda_K & \kappa_{z,V} & -\kappa_{z,K} \\ \lambda_K & \lambda_M & -\kappa_{z,K} & \kappa_{z,M} \end{bmatrix} \begin{bmatrix} w_E \\ \varphi_E L \\ v_E \\ \psi_E L \end{bmatrix} \tag{4.66}$$

The equilibrium condition of the form $\vec{b} = \sum \boldsymbol{H}_{(e)}^T \cdot \vec{b}_{(e)}$ follows to:

$$\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} F = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} V_{zE} \\ M_{yE}/L \\ V_{yE} \\ M_{zE}/L \end{bmatrix} \tag{4.67}$$

The system equation of the form $\vec{b} = \boldsymbol{K} \cdot \vec{s}$ is determined through the combination of (4.65), (4.66) and (4.67).

$$\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} F_z = \frac{EI_y}{L^3} \begin{bmatrix} \kappa_{y,V} & \kappa_{y,K} & -\lambda_V & \lambda_K \\ \kappa_{y,K} & \kappa_{y,M} & -\lambda_K & \lambda_M \\ -\lambda_V & -\lambda_K & \kappa_{z,V} & -\kappa_{z,K} \\ \lambda_K & \lambda_M & -\kappa_{z,K} & \kappa_{z,M} \end{bmatrix} \begin{bmatrix} s_1 \\ s_2 \\ s_3 \\ s_4 \end{bmatrix} \tag{4.68}$$

To find the critical beam stability identification number, the determinant of the stiffness matrix must be inspected. For $\omega = 0$, the graph of the determinant is illustrated below. Evidently, the critical value is $\varepsilon_{cr}(\alpha, \omega = 0) = \frac{\pi}{2}$. Using (4.60), all other critical values are henceforth:

$$\varepsilon_{cr}(\alpha, \omega) = \frac{\pi}{2} \cdot \sqrt{\frac{1+\alpha}{2} - \sqrt{\left(\frac{1+\alpha}{2}\right)^2 + \omega^2}} \tag{4.69}$$

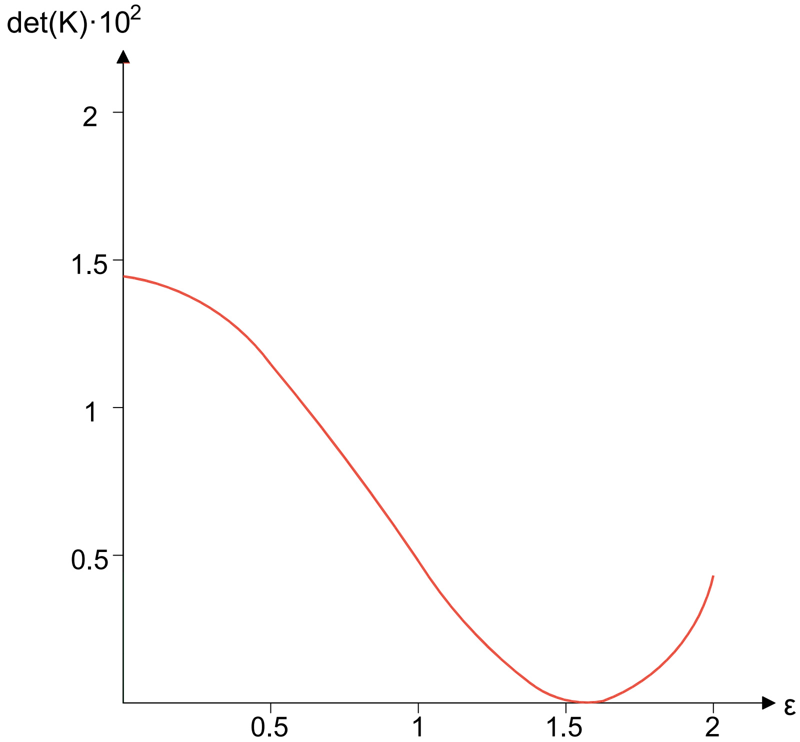


Fig. 4-23 Determinant of $\boldsymbol{K}$ for $\alpha = 1, \omega = 0$ [5]

The described system is modeled in the software for a further, less generalized, analysis. The beam-column is given a length of 1m and a DIN 1027:2004-04 Z140 section [7]. The steel is S235. The values for P and F_z are $200kN$ and $1kN$, respectively.

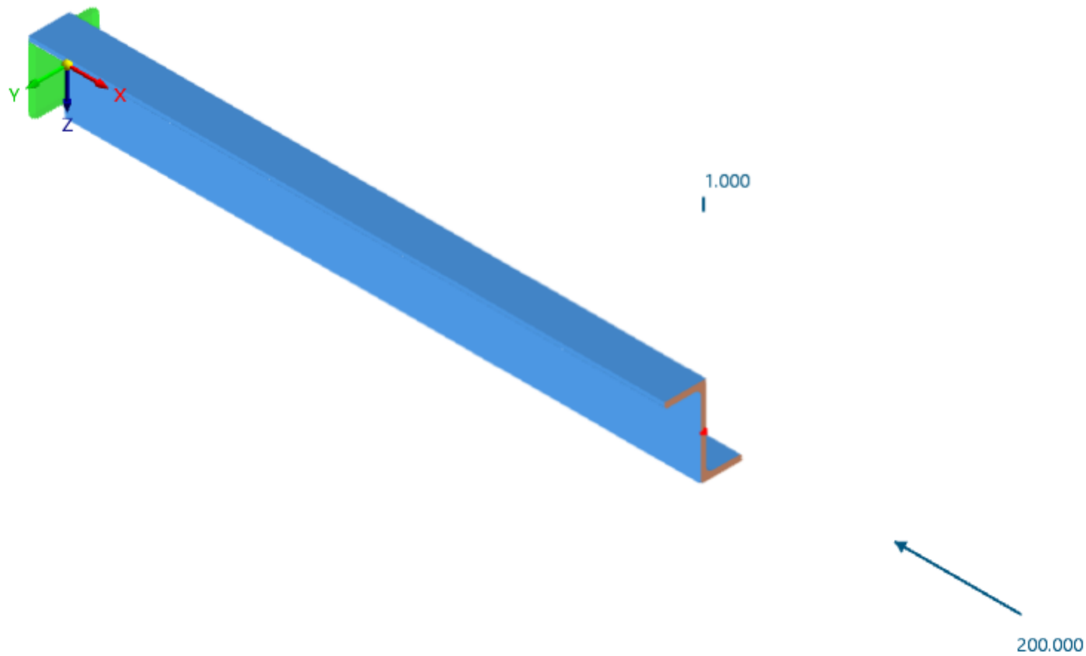


Fig. 4-24 Fixed-Free Column in RFEM [5]

The given data result in the following parameters.

$$\alpha = \frac{148cm^4}{676cm^4} = 0.2189 \tag{4.70}$$

$$\omega = \frac{239cm^4}{676cm^4} = 0.3536 \tag{4.71}$$

$$E = 21000 \frac{kN}{cm^2} \tag{4.72}$$

$$L = 100cm \tag{4.73}$$

$$F_z = 1kN \tag{4.74}$$

$$P = 200kN \tag{4.75}$$

Using the relation from (4.69) and the above values, the critical beam identification value becomes:

$$\varepsilon_{cr}(\alpha = 0.2189, \omega = 0.3536) = 0.4515 \tag{4.76}$$

The critical load is then:

$$P_{cr} = \frac{0.4516^2 \cdot 21000 \frac{kN}{cm^2} \cdot 676cm^4}{(100cm)^2} = 289.5kN \tag{4.77}$$

This value coincides with the value given by the software:

$$P_{cr,RSTAB} = 292.3kN \tag{4.78}$$

With the given load of $P = 200kN$, the present value for ε is:

$$\varepsilon = \sqrt{\frac{200kN \cdot (100cm)^2}{21000 \frac{kN}{cm^2} \cdot 676cm^4}} = 0.375 \tag{4.79}$$

For the present value, the system equation from (4.68) becomes:

$$\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} F_z = \frac{EI_y}{L^3} \begin{bmatrix} 11.831 & 5.986 & -4.243 & 2.121 \\ 5.986 & 3.981 & -2.121 & 1.414 \\ -4.243 & -2.121 & 2.458 & -1.299 \\ 2.121 & 1.414 & -1.299 & 0.857 \end{bmatrix} \begin{bmatrix} s_1 \\ s_2 \\ s_3 \\ s_4 \end{bmatrix} \tag{4.80}$$

By inverting (4.80), the displacement vector becomes:

$$\begin{bmatrix} s_1 \\ s_2 \\ s_3 \\ s_4 \end{bmatrix} = \begin{bmatrix} 1.909 \\ -2.937 \\ 4.147 \\ 6.407 \end{bmatrix} \frac{F_z L^3}{EI_y} \tag{4.81}$$

The deflection of a cantilever induce by an isolated load were previously determined in (3.45) on the basis of First Order Theory. Using the given values, these are then:

$$\begin{bmatrix} w_z^I(1) \\ w_y^I(1) \end{bmatrix} = \begin{bmatrix} 0.55 \\ 0.88 \end{bmatrix} mm \tag{4.82}$$

On the basis of Second Order Theory, the deflections are given by (4.81), and follow to:

$$\begin{bmatrix} w_z^{II}(1) \\ w_y^{II}(1) \end{bmatrix} = \begin{bmatrix} 1.35 \\ 2.92 \end{bmatrix} mm \tag{4.83}$$

Both of these results align with the results given by the software.

$$\begin{bmatrix} w_z^I(1) \\ w_y^I(1) \end{bmatrix}_{RSTAB} = \begin{bmatrix} 0.5 \\ 0.9 \end{bmatrix} mm \tag{4.84}$$

$$\begin{bmatrix} w_z^{II}(1) \\ w_y^{II}(1) \end{bmatrix}_{RSTAB} = \begin{bmatrix} 1.3 \\ 2.9 \end{bmatrix} mm \tag{4.85}$$

For a variable value of P, the deformations $w_z^{II}(1)$ and $w_y^{II}(1)$ can be plotted. The figure below shows these deformations as a function of ε. It can be observed that there exists a vertical asymptote for both functions located at $\varepsilon = \varepsilon_{cr}$. This corresponds to the theoretical buckling condition in which the deformations approach infinity.

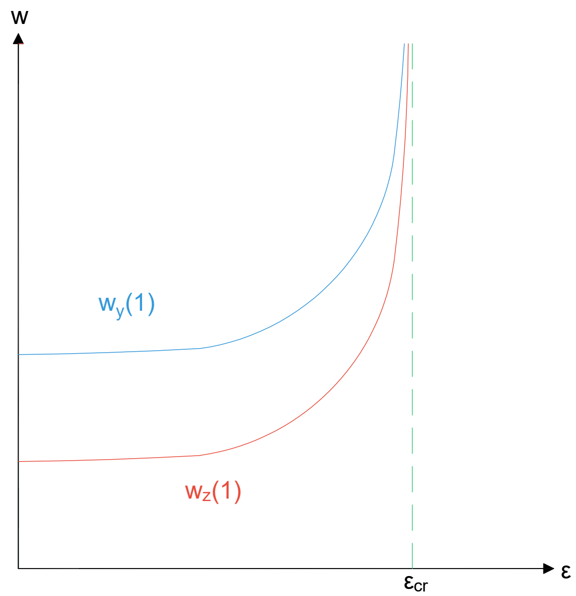


Fig. 4-25 $w_z^{II}(1)$ and $w_y^{II}(1)$ as a function of ε_{cr} [5]

The bending moments based on Second Order Theory are given by (4.55). Using the known displacements for $P = 200kN$ in (4.81), the bending moments are expressed through the following.

$$\begin{bmatrix} M_y \\ M_z \end{bmatrix} = \frac{EI_y}{L^2} \begin{bmatrix} \mu^{II}_{w_E} & \mu^{II}_{\varphi_E} & \mu^{II}_{wv_E} & \mu^{II}_{\varphi\psi_E} \\ -\mu^{II}_{wv_E} & \mu^{II}_{\varphi\psi_E} & \mu^{II}_{v_E} & \mu^{II}_{\psi_E} \end{bmatrix} \begin{bmatrix} 1.909 \\ -2.937 \\ 4.147 \\ 6.407 \end{bmatrix} \frac{F_z L^3}{EI_y} \tag{4.86}$$

These result in the following moment diagrams. They match the diagrams given by the software.

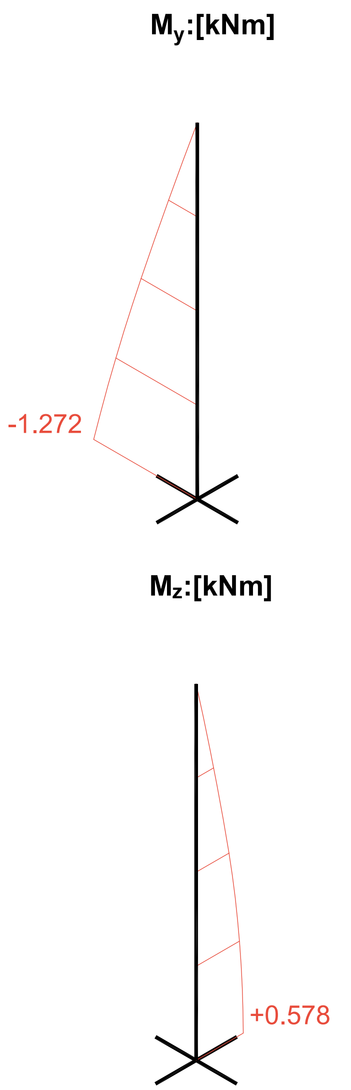


Fig. 4-26 Moment Diagrams [5]

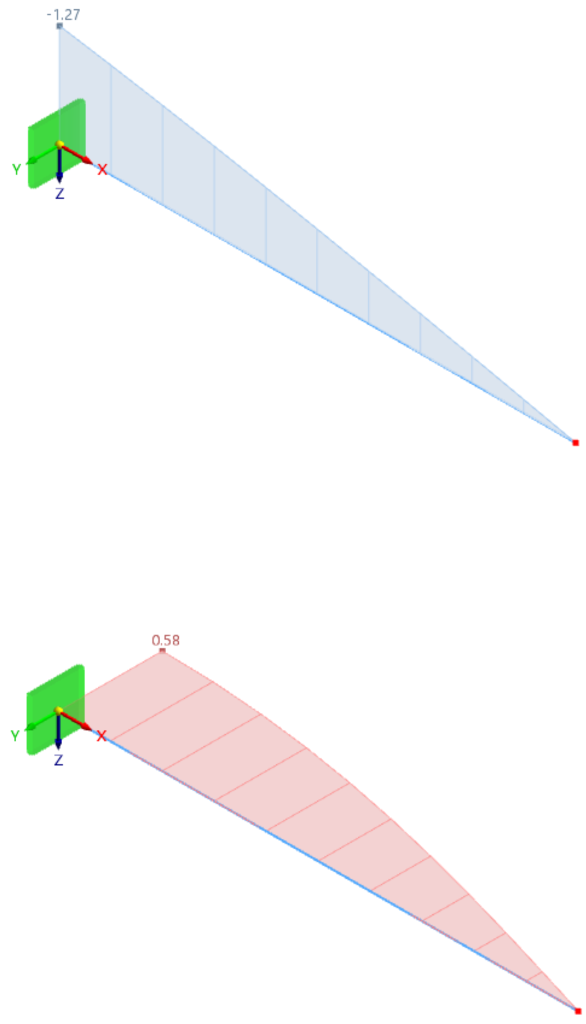


Fig. 4-27 Moment Diagrams in RSTAB [5]

Since Second Order Theory represents the equilibrium condition of the deformed system, the moments shown in Fig. 4-26 can be computed and subsequently checked using the equilibria of moments. The equilibria in the deformed state are:

$$\begin{bmatrix} M_y(0) \\ M_z(0) \end{bmatrix} = \begin{bmatrix} -L & -w_z^{II}(1) \\ 0 & w_y^{II}(1) \end{bmatrix} \cdot \begin{bmatrix} F \\ P \end{bmatrix} \tag{4.87}$$

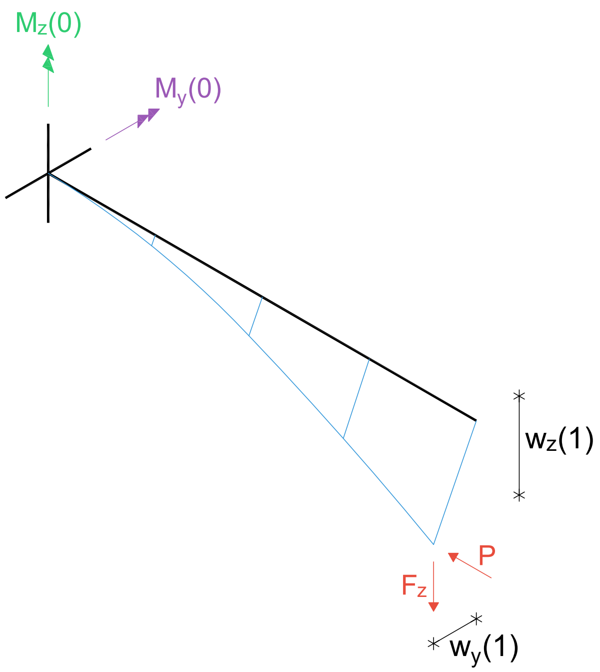


Fig. 4-28 Equilibrium Condition in the Deformed State [5]

Inserting the values yields similar moments as shown in Fig. 4-26.

$$\begin{bmatrix} M_y(0) \\ M_z(0) \end{bmatrix} = \begin{bmatrix} -1m & -1.35mm \\ 0 & 2.92mm \end{bmatrix} \cdot \begin{bmatrix} 1kN \\ 200kN \end{bmatrix} = \begin{bmatrix} -1.27 \\ +0.584 \end{bmatrix} kNm \tag{4.88}$$

4.4.3 Frame

The following system consists of three beams that make up a freestanding frame. Each member has the same section; a DIN EN 10056-1:1998-10 L150x150x10 [7]. The steel is S235. The frame has a height of $2m$ and a length of $4m$. As seen in the figure below, the beam-columns are subjected to the same axial force P.

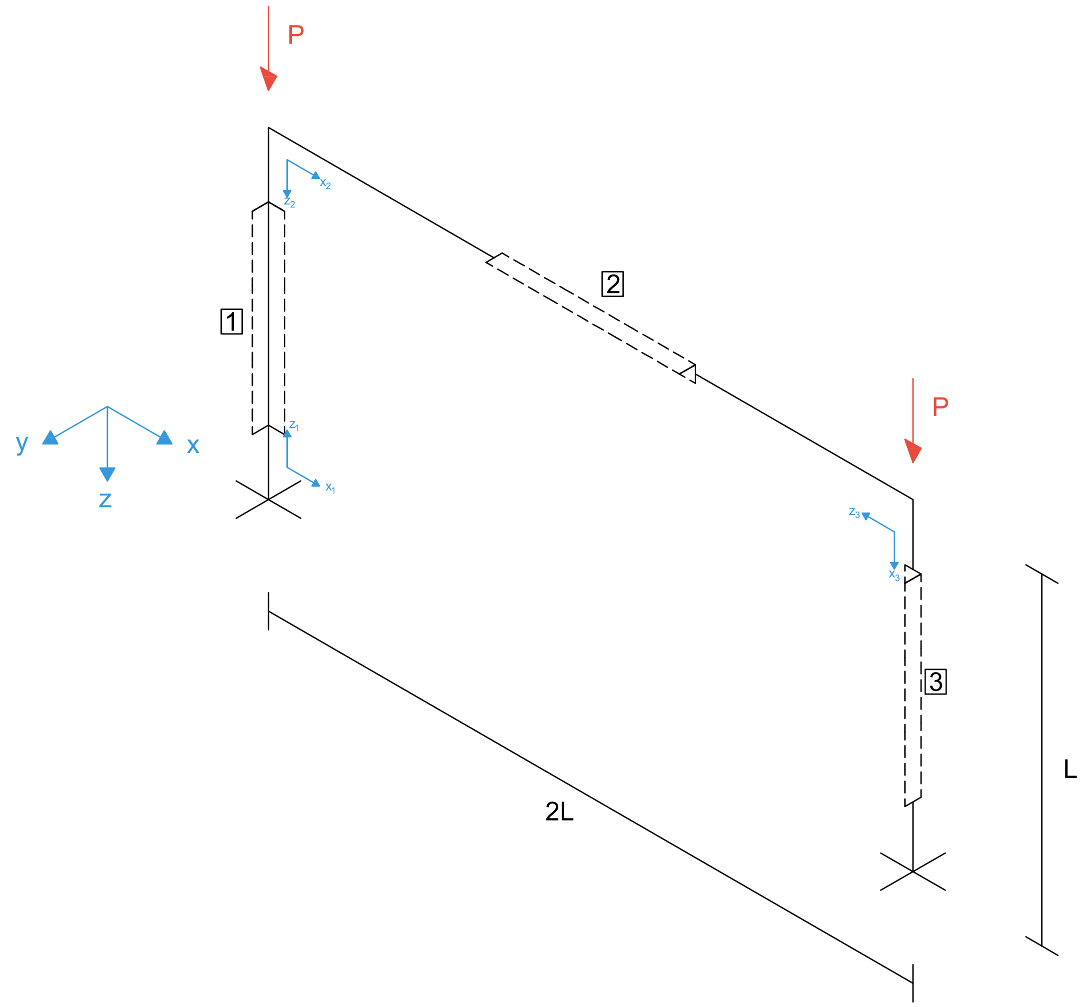


Fig. 4-29 Freestanding Frame [5]

Within the scope of this thesis, this analysis includes the following approximations. Firstly, the members are assumed to be axially rigid. Secondly, the additional stiffnesses attributed to torsion are neglected. This is because on the one hand, the effects of Second Order Theory on torsion are not discussed in this thesis. On the other hand, the additional stiffnesses from torsion are comparably small, since the cross-section is weak in torsion.

With the above made approximations, the kinematic indeterminacy of the system is nine. These nine distinct degrees of freedom are indicated below.

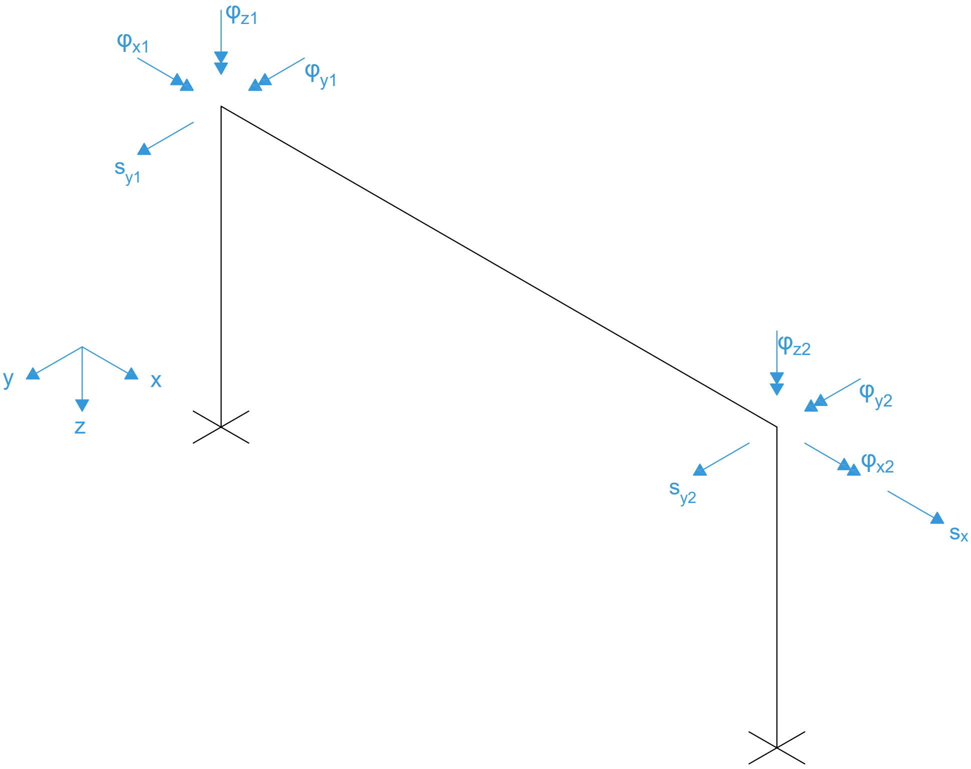
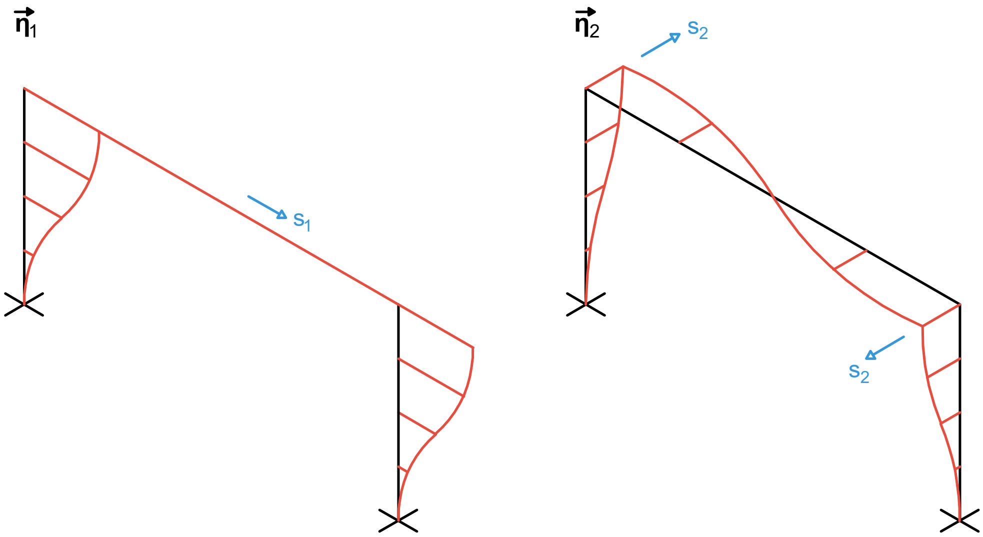
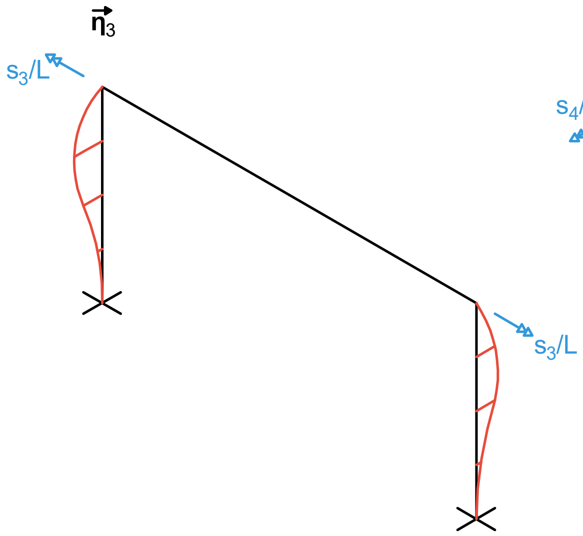
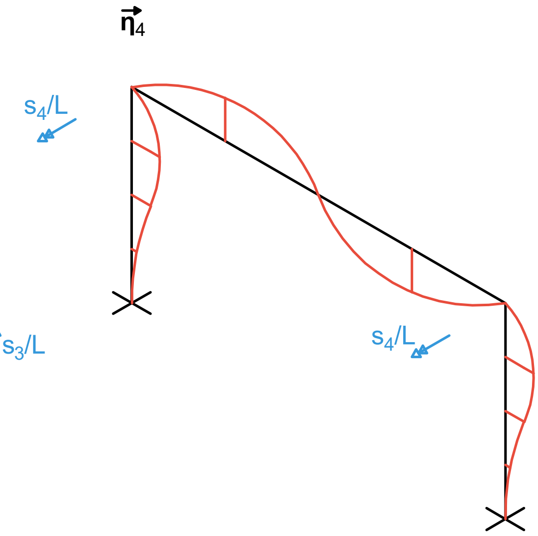


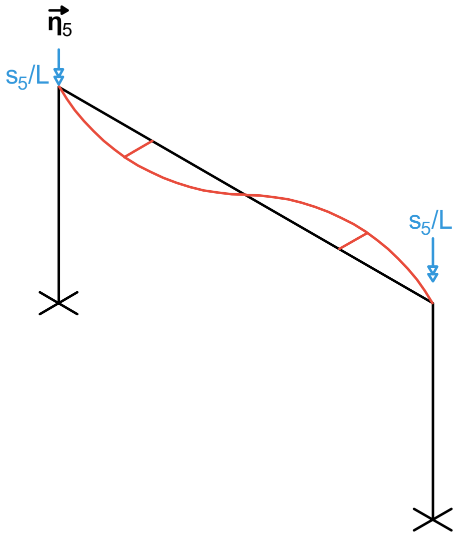
Fig. 4-30 Nine Distinct Degrees of Freedom of the Frame [5]

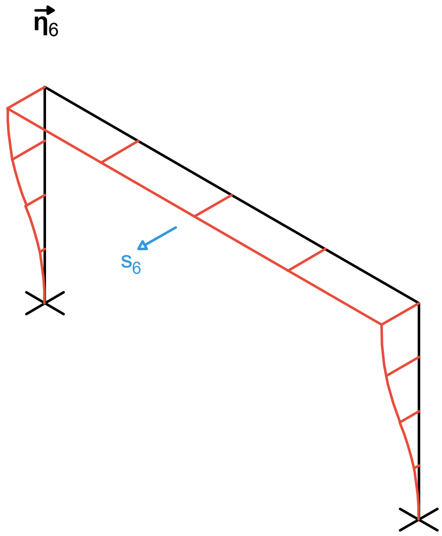
For a simpler computation, the degrees of freedom presented above are combined linearly to form symmetric and antisymmetric degrees of freedom. These are illustrated below. There exist five antisymmetric degrees of freedom, η_1 to η_5, and four symmetric degrees of freedom, η_6 to η_9.

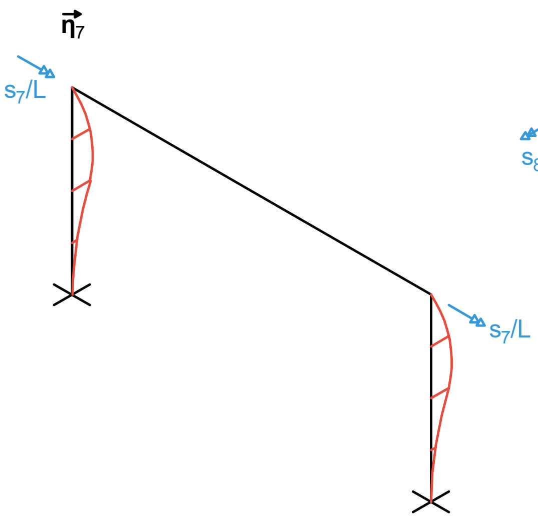


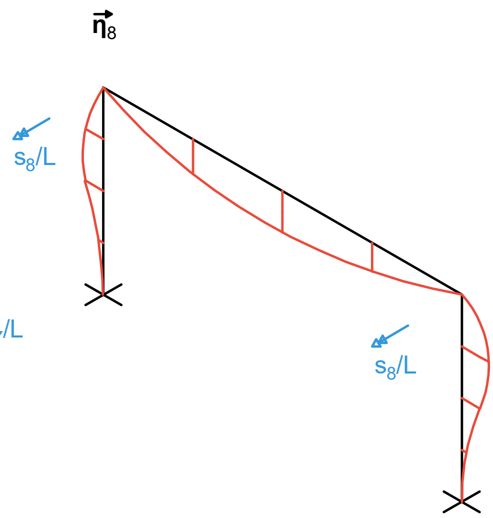

$\vec{\eta}_3$
s_3/L
s_3/L


$\vec{\eta}_4$
s_4/L
s_4/L


$\vec{\eta}_5$
s_5/L
s_5/L


$\vec{\eta}_6$
s_6


$\vec{\eta}_7$
s_7/L
s_7/L


$\vec{\eta}_8$
s_8/L
s_8/L

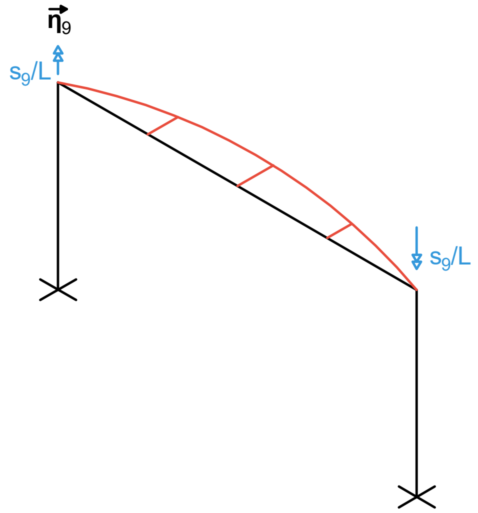


Fig. 4-31 Symmetric and Antisymmetric Degrees of Freedom [5]

The kinematic relationships for each member respectively are:

$$\begin{bmatrix} w_E \\ \varphi_E L \\ v_E \\ \psi_E L \end{bmatrix}_{(1)} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} s_1 \\ s_2 \\ s_3 \\ s_4 \\ s_5 \\ s_6 \\ s_7 \\ s_8 \\ s_9 \end{bmatrix} \tag{4.89}$$

$$\begin{bmatrix} \varphi_A 2L \\ \varphi_E 2L \\ v_A \\ \psi_A 2L \\ v_E \\ \psi_E 2L \end{bmatrix}_{(2)} = \begin{bmatrix} 0 & 0 & 0 & 2 & 0 & 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 & 0 & 2 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & -2 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} s_1 \\ s_2 \\ s_3 \\ s_4 \\ s_5 \\ s_6 \\ s_7 \\ s_8 \\ s_9 \end{bmatrix} \tag{4.90}$$

$$\begin{bmatrix} w_A \\ \varphi_A L \\ v_A \\ \psi_A L \end{bmatrix}_{(3)} = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} s_1 \\ s_2 \\ s_3 \\ s_4 \\ s_5 \\ s_6 \\ s_7 \\ s_8 \\ s_9 \end{bmatrix} \tag{4.91}$$

For members 1 and 3 the constitutive relationship is of Second Order Theory and stems from (4.50). Since member 2 experiences no axial force, its constitutive relationship is of First Order Theory and stems from (3.36).

$$\begin{bmatrix} V_{zE} \\ M_{yE}/L \\ V_{yE} \\ M_{zE}/L \end{bmatrix}_{(1)} = \frac{EI_y}{L^3} \begin{bmatrix} \kappa_{y,V} & \kappa_{y,K} & -\lambda_V & \lambda_K \\ \kappa_{y,K} & \kappa_{y,M} & -\lambda_K & \lambda_V \\ -\lambda_V & -\lambda_K & \kappa_{z,V} & -\kappa_{z,K} \\ \lambda_K & \lambda_V & -\kappa_{z,K} & \kappa_{z,M} \end{bmatrix} \begin{bmatrix} w_E \\ \varphi_E L \\ v_E \\ \psi_E L \end{bmatrix}_{(1)} \tag{4.92}$$

$$\begin{bmatrix} M_{yE}/2L \\ M_{yE}/2L \\ V_{yE} \\ M_{zE}/2L \\ V_{yE} \\ M_{zE}/2L \end{bmatrix}_{(2)} = \frac{EI_y}{8 \cdot L^3} \begin{bmatrix} 4 & 2 & 6\omega & 4\omega & -6\omega & 2\omega \\ 2 & 4 & 6\omega & 2\omega & -6\omega & 4\omega \\ 6\omega & 6\omega & 12\alpha & 6\alpha & -12\alpha & 6\alpha \\ 4\omega & 2\omega & 6\alpha & 4\alpha & -6\alpha & 2\alpha \\ -6\omega & -6\omega & -12\alpha & -6\alpha & 12\alpha & -6\alpha \\ 2\omega & 4\omega & 6\alpha & 2\alpha & -6\alpha & 4\alpha \end{bmatrix} \begin{bmatrix} \varphi_A 2L \\ \varphi_E 2L \\ v_A \\ \psi_A 2L \\ v_E \\ \psi_E 2L \end{bmatrix}_{(2)} \tag{4.93}$$

$$\begin{bmatrix} V_{zA} \\ M_{yA}/L \\ V_{yA} \\ M_{zA}/L \end{bmatrix}_{(3)} = \frac{EI_y}{L^3} \begin{bmatrix} \kappa_{y,V} & -\kappa_{y,K} & -\lambda_V & -\lambda_K \\ -\kappa_{y,K} & \kappa_{y,M} & \lambda_K & \lambda_V \\ -\lambda_V & \lambda_K & \kappa_{z,V} & \kappa_{z,K} \\ -\lambda_K & \lambda_V & \kappa_{z,K} & \kappa_{z,M} \end{bmatrix} \begin{bmatrix} w_A \\ \varphi_A L \\ v_A \\ \psi_A L \end{bmatrix}_{(3)} \tag{4.94}$$

Using (1.13) and the relations (4.89) through (4.94), the system's stiffness matrix becomes:

$$\boldsymbol{K} = \frac{EI_y}{L^3} \cdot \begin{bmatrix} \boldsymbol{K}_{asym} & 0 \\ 0 & \boldsymbol{K}_{sym} \end{bmatrix} \tag{4.95}$$

Where:

$$\boldsymbol{K}_{asym} = 2 \cdot \begin{bmatrix} \kappa_{y,V} & \lambda_V & -\lambda_K & \kappa_{y,K} & 0 \\ \lambda_V & \kappa_{z,V} + 3\alpha & -\kappa_{z,K} & \lambda_K - 3\omega & -3\alpha \\ -\lambda_K & -\kappa_{z,K} & \kappa_{z,M} & -\lambda_M & 0 \\ \kappa_{y,K} & \lambda_K - 3\omega & -\lambda_M & \kappa_{y,M} + 3\alpha & 3\omega \\ 0 & -3\alpha & 0 & 3\omega & 3\alpha \end{bmatrix} \tag{4.96}$$

$$\boldsymbol{K}_{sym} = 2 \cdot \begin{bmatrix} \kappa_{z,V} & -\kappa_{z,K} & \lambda_K & 0 \\ -\kappa_{z,K} & \kappa_{z,M} & -\lambda_M & 0 \\ \lambda_K & -\lambda_M & \kappa_{y,M} + 1 & \omega \\ 0 & 0 & \omega & \alpha \end{bmatrix} \tag{4.97}$$

When the determinant of the stiffness matrix becomes zero the system buckles. Due to the chosen degrees of freedom, the stiffness matrix consists of two submatrices. They describe the antisymmetric and symmetric deformations of the system. Illustrated below are the determinants of the submatrices as a function of ε. It can be observed that the determinant corresponding to the antisymmetric deformations always has the lowest ε -value for when it intercepts the horizontal axis. This point is the entire system's critical beam stability value and demonstrates that when the system buckles, it buckles asymmetrically.

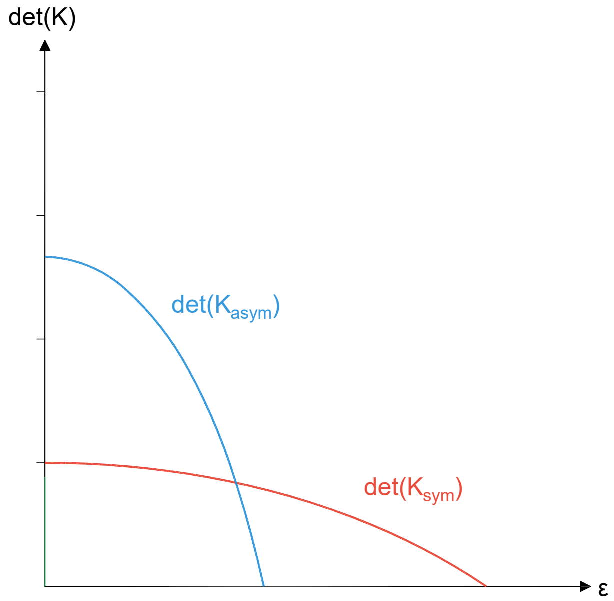


Fig. 4-32 Determinants $det(\boldsymbol{K}_{asym})$ and $det(\boldsymbol{K}_{sym})$ [5]

For the chosen sections, the necessary parameters become:

$$\alpha = 1 \tag{4.98}$$

$$\omega = \frac{366.55cm^4}{624.05cm^4} = 0.5874 \tag{4.99}$$

$$E = 21000\frac{kN}{cm^2} \tag{4.100}$$

$$L = 200cm \tag{4.101}$$

For these values, the critical beam stability identification number follows to:

$$\varepsilon_{cr} = 1.337 \tag{4.102}$$

The critical axial force is thus:

$$P_{cr} = (1.337)^2 \cdot \frac{21000\frac{kN}{cm^2} \cdot 624.05cm^4}{(200cm)^2} = 586kN \tag{4.103}$$

This value matches the result given by the software when the same approximations discussed earlier are made.

$$P_{cr_{RSTAB}} = 590kN \tag{4.104}$$

In fact, when the approximations are not made, the results do not change significantly.

$$P_{cr_{RSTAB,exact}} = 589kN \tag{4.105}$$

It has been previously stated that when the system buckles, it buckles asymmetrically. The buckling shape corresponds to the eigenvector of the stiffness matrix with the lowest eigenvalue. For $\varepsilon_{cr} = 1.337$, the stiffness submatrix $\boldsymbol{K}_{asym}$ takes the following values.

$$\boldsymbol{K}_{asym}(\varepsilon = 1.337) \approx \begin{bmatrix} 19.696 & 14.106 & -7.052 & 11.636 & 0 \\ 14.106 & 25.696 & -11.636 & 3.528 & -6 \\ -7.052 & -11.636 & 7.504 & -4.71 & 0 \\ 11.636 & 3.528 & -4.71 & 13.504 & 3.524 \\ 0 & -6 & 0 & 3.524 & 6 \end{bmatrix} \tag{4.106}$$

The eigenvector with the lowest eigenvalue, in this case -0.002, expresses the buckling shape of the system. Note that theoretically the eigenvalue should be zero. The critical eigenvector of the matrix is then:

$$\vec{\boldsymbol{v}}_{cr} \approx \begin{bmatrix} -0.660 \\ 1.354 \\ 1.856 \\ 0.602 \\ 1 \end{bmatrix} \tag{4.107}$$

The eigenvector is written in the vector space where the basis is spun by the degrees of freedom η_1 to η_5. The graphical representation of the buckling shape, denoted by $\vec{\boldsymbol{\eta}}_{cr}$, is a linear combination of the first five degrees of freedom shown in Fig. 4-30.

$$\vec{\boldsymbol{\eta}}_{cr} \approx -0.660 \cdot \vec{\boldsymbol{\eta}}_1 + 1.354 \cdot \vec{\boldsymbol{\eta}}_2 + 1.856 \cdot \vec{\boldsymbol{\eta}}_3 + 0.602 \cdot \vec{\boldsymbol{\eta}}_4 + \vec{\boldsymbol{\eta}}_5 \tag{4.108}$$

With

$\vec{\boldsymbol{\eta}}_{cr}$ ... Buckling shape

The system's buckling shape is illustrated below. Additionally, the buckling shape generated by the software is given as a comparison.

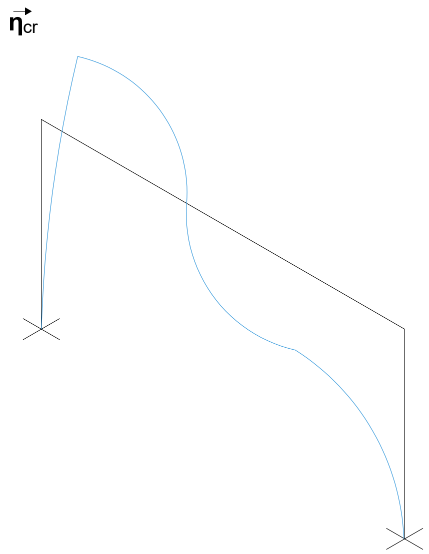


Fig. 4-33 Buckling Shape [5]

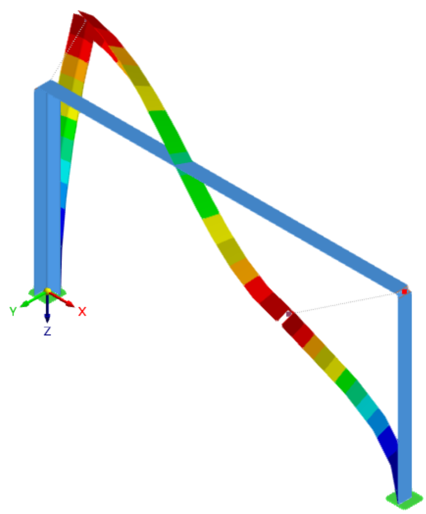


Fig. 4-34 Buckling Shape by RSTAB [5]

5 Conclusion

This thesis explored how unsymmetrical bending can be expressed through the Displacement Method. After providing the necessary background of theories and definitions, this thesis developed and solved systems of differential equations pertaining to First and Second Order Analysis. Using these solutions, the constitutive relationships that describe unsymmetrical bending were derived. Additionally, these relationships were applied to several exemplary systems where unsymmetrical bending occurs or can be applied. Furthermore, the results of these analyses were compared with the established finite element software RSTAB 9. First order analyses have an accuracy of 100% when compared with the results of the software. Due to the discrepancy between the exact constitutive relationships and the approximate finite element analysis, the results of second order analyses led to a minimum accuracy of 96% and a maximum accuracy of 100%.

The derived relationships correspond to an exact element theory and provide results of higher accuracy than finite element theory. That stated, the relations are only accurate for elastic prismatic beams without shear deformations or torsional loads. While this thesis did not create new theories of bending, it examined existing theories of symmetrical bending and expanded them to unsymmetrical bending. The goal of deriving closed-form mathematical relations to analyze statically determinate and indeterminate beams subjected to unsymmetrical bending using the Displacement Method has been accomplished.

References

[1] W. H. J. S. W. A. W. Dietmar Gross, Technische Mechanik 2 Elastostatik 14. Auflage, Berlin: Springer Vieweg Berlin, Heidelberg, 2021.

[2] W. H. P. W. Dietmar Gross, Technische Mechanik 4, Berlin: Springer Vieweg Berlin, Heidelberg, 2023.

[3] K. K. X. Z. Xiuzhang He, "Closed-form design solutions for parallelogram hollow structural sections under bending scenarios," *Journal of Constructional Steel Research,* vol. 207, 2023.

[4] O. L. Igor A. Karnovsky, Advanced Methods of Structural Analysis, Springer New York, NY, 2010.

[5] B. Jugel, *self-created image.*

[6] Dlubal Software, "RSTAB 9: Structural Analysis Software for Structural Planners & Engineers," Dlubal Software, [Online]. Available: https://www.dlubal.com/en/products/rstab-beam-structures/rstab-9/what-is-rstab. [Accessed 1 April 2024].

[7] Dlubal Software, "Cross-Section Properties & Analysis | Dlubal Software," Dlubal Software , [Online]. Available: https://www.dlubal.com/en/cross-section-properties/. [Accessed 1 April 2024].

[8] Dlubal Software, "RSECTION 1 | Section Properties and Stress Analysis," Dlubal Software, [Online]. Available: https://www.dlubal.com/en/products/querschnittswerte-programme/rsection. [Accessed 1 April 2024].

[9] P. Dawkins, " Differential Equations - Solutions to Systems," [Online]. Available: https://tutorial.math.lamar.edu/Classes/DE/SolutionsToSystems.aspx. [Accessed 1 April 2024].

[10] P. Dawkins, "Differential Equations - Complex Roots," [Online]. Available: https://tutorial.math.lamar.edu/Classes/DE/ComplexRoots.aspx. [Accessed 1 April 2024].

Attachment A

The First Order amplitude functions η^I from 3.2.4 are defined as follows.

$$\eta^I_{w_A} := 2\xi^3 - 3\xi^2 + 1$$

$$\eta^I_{\varphi_A} := -\xi^3 + 2\xi^2 - \xi$$

$$\eta^I_{w_E} := -2\xi^3 + 3\xi^2$$

$$\eta^I_{\varphi_E} := -\xi^3 + \xi^2$$

$$\eta^I_{v_A} := 2\xi^3 - 3\xi^2 + 1$$

$$\eta^I_{\psi_A} := \xi^3 - 2\xi^2 + \xi$$

$$\eta^I_{v_E} := -2\xi^3 + 3\xi^2$$

$$\eta^I_{\psi_E} := \xi^3 - \xi^2$$

$$\eta^I_{\rho} := \xi^4 - 2\xi^3 + \xi^2$$

Attachment B

The First Order amplitude functions η^I from 3.2.5 are defined as follows.

$$\mu_{w_A}^I := -12\xi + 6$$

$$\mu_{\varphi_A}^I := 6\xi - 4$$

$$\mu_{w_E}^I := 12\xi - 6$$

$$\mu_{\varphi_E}^I := 6\xi - 2$$

$$\mu_{v_A}^I := \alpha \cdot (12\xi - 6)$$

$$\mu_{\psi_A}^I := \alpha \cdot (6\xi - 4)$$

$$\mu_{v_E}^I := \alpha \cdot (-12\xi + 6)$$

$$\mu_{\psi_E}^I := \alpha \cdot (6\xi - 2)$$

$$\mu_{wv_A}^I := \omega \cdot (12\xi - 6)$$

$$\mu_{\varphi\psi_A}^I := \omega \cdot (6\xi - 4)$$

$$\mu_{wv_E}^I := \omega \cdot (-12\xi + 6)$$

$$\mu_{\varphi\psi_E}^I := \omega \cdot (6\xi - 2)$$

$$\mu_{\rho}^I := \frac{-6\xi^2 + 6\xi - 1}{12}$$

Attachment C

The stiffness coefficients for the constitutive relationship of Second Order Theory, presented in (4.51), (4.52) and (4.53), are defined through the following expressions.

For $\boldsymbol{K}_y^{II}$:

$$\kappa_{y,V} := \frac{\varepsilon^4 \cdot \Psi_{V,1} + \varepsilon^3 \cdot \Psi_{V,2}}{\Psi_0}$$

$$\kappa_{y,M} := \frac{\varepsilon^3 \cdot \Psi_{M,1} + \varepsilon^2 \cdot \Psi_{M,2} + \varepsilon \cdot \Psi_{M,3}}{\Psi_0}$$

$$\kappa_{y,K} := \frac{\varepsilon^3 \cdot \Psi_{K,1} + \varepsilon^2 \cdot \Psi_{K,2}}{\Psi_0}$$

$$\kappa_{y,M}^* := \kappa_{y,K} - \kappa_{y,M}$$

Where:

$$\Psi_0 \coloneqq [\gamma_2 - \gamma_1] \cdot [\bar{s}_1\beta_1\varepsilon - 2(1 - \bar{c}_1)] \cdot [\bar{s}_2\beta_2\varepsilon - 2(1 - \bar{c}_2)]$$

$$\Psi_{V,1} \coloneqq \beta_1\beta_2\bar{s}_1\bar{s}_2 \cdot [\beta_1^2(\omega - \gamma_1) - \beta_2^2(\omega - \gamma_2)]$$

$$\Psi_{V,2} \coloneqq \beta_1^3\bar{s}_1 \cdot 2(1 - \bar{c}_1)(\omega - \gamma_1) - \beta_2^3\bar{s}_2 \cdot 2(1 - \bar{c}_2)(\omega - \gamma_2)$$

$$\Psi_{M,1} \coloneqq \beta_1\beta_2 \cdot [\beta_1\bar{c}_1\bar{s}_2(\omega - \gamma_1) - \beta_2\bar{c}_2\bar{s}_1(\omega - \gamma_2)]$$

$$\Psi_{M,2} \coloneqq \beta_1^2\bar{c}_1 \cdot 2(\bar{c}_2 - 1)(\omega - \gamma_1) - \beta_2^2\bar{c}_2 \cdot 2(\bar{c}_1 - 1)(\omega - \gamma_2) + \beta_1\beta_2\bar{s}_1\bar{s}_2 \cdot [\gamma_1 - \gamma_2]$$

$$\Psi_{M,3} \coloneqq \beta_1\bar{s}_1 \cdot 2(1 - \bar{c}_2)(\omega - \gamma_1) - \beta_2 s_2 \cdot 2(1 - \bar{c}_1)(\omega - \gamma_2)$$

$$\Psi_{K,1} \coloneqq \beta_1\beta_2 \cdot [\beta_1\bar{s}_2 \cdot (\bar{c}_1 - 1)(\omega - \gamma_1) - \beta_2\bar{s}_1 \cdot (\bar{c}_2 - 1)(\omega - \gamma_2)]$$

$$\Psi_{K,2} \coloneqq 2 \cdot (\bar{c}_1 - 1)(1 - \bar{c}_2)[\beta_1^2(\gamma_1 - \omega) - \beta_2^2(\gamma_2 - \omega)]$$

For $\boldsymbol{K}_z^{II}$:

$$\kappa_{z,V} := \frac{\varepsilon^4 \cdot \Phi_{V,1} + \varepsilon^3 \cdot \Phi_{V,2}}{\Phi_0}$$

$$\kappa_{z,M} := \frac{\varepsilon^3 \cdot \Phi_{M,1} + \varepsilon^2 \cdot \Phi_{M,2} + \varepsilon \cdot \Phi_{M,3}}{\Phi_0}$$

$$\kappa_{z,K} := \frac{\varepsilon^3 \cdot \Phi_{K,1} + \varepsilon^2 \cdot \Phi_{K,2}}{\Phi_0}$$

$$\kappa_{z,M}^* := \kappa_{z,K} - \kappa_{z,M}$$

Where:

$$\Phi_0 \coloneqq [\gamma_2 - \gamma_1] \cdot [\bar{s}_1\beta_1\varepsilon - 2(1-\bar{c}_1)] \cdot [\bar{s}_2\beta_2\varepsilon - 2(1-\bar{c}_2)]$$

$$\Phi_{V,1} \coloneqq \beta_1\beta_2\bar{s}_1\bar{s}_2 \cdot [\beta_1^2\gamma_2(\gamma_1\omega - \alpha) - \beta_2^2\gamma_1(\gamma_2\omega - \alpha)]$$

$$\Phi_{V,2} \coloneqq \beta_1^3\gamma_2\bar{s}_1 \cdot 2(\alpha - \gamma_1\omega)(1-\bar{c}_2) - \beta_2^3\gamma_1\bar{s}_2 \cdot 2(\alpha - \gamma_2\omega)(1-\bar{c}_1)$$

$$\Phi_{M,1} \coloneqq \beta_1\beta_2[\beta_1\bar{c}_1\gamma_2\bar{s}_2(\alpha - \gamma_1\omega) - \beta_2\bar{c}_2\gamma_1\bar{s}_1(\alpha - \gamma_2\omega)]$$

$$\Phi_{M,2} \coloneqq \beta_1^2\bar{c}_1\gamma_2 \cdot 2(\gamma_1\omega - \alpha)(1-\bar{c}_2) - \beta_2^2\bar{c}_2\gamma_1 \cdot 2(\gamma_2\omega - \alpha)(1-\bar{c}_1)$$
$$+\alpha\beta_1\beta_2\bar{s}_1\bar{s}_2 \cdot [\gamma_1 - \gamma_2]$$

$$\Phi_{M,3} \coloneqq \beta_1\gamma_2\bar{s}_1 \cdot 2(\gamma_1\omega - \alpha)(\bar{c}_2 - 1) - \beta_2\gamma_1\bar{s}_2 \cdot 2(\gamma_2\omega - \alpha)(\bar{c}_1 - 1)$$

$$\Phi_{K,1} \coloneqq \beta_1\beta_2[\beta_1\gamma_2\bar{s}_2(\gamma_1\omega - \alpha)(1-\bar{c}_1) - \beta_2\gamma_1\bar{s}_1(\gamma_2\omega - \alpha)(1-\bar{c}_2)]$$

$$\Phi_{K,2} \coloneqq (\bar{c}_1 - 1) \cdot (1-\bar{c}_2) \cdot [\beta_1^2\gamma_2 \cdot 2(\gamma_1\omega - \alpha) - \beta_2^2\gamma_1 \cdot 2(\gamma_2\omega - \alpha)]$$

For $\boldsymbol{K_{yz}^{II}}$:

$$\lambda_V \coloneqq \frac{\varepsilon^4 \cdot \Omega_{V,1} + \varepsilon^3 \cdot \Omega_{V,2}}{\Omega_0}$$

$$\lambda_M \coloneqq \frac{\varepsilon^3 \cdot \Omega_{M,1} + \varepsilon^2 \cdot \Omega_{M,2} + \varepsilon \cdot \Omega_{M,3}}{\Omega_0}$$

$$\lambda_K \coloneqq \frac{\varepsilon^3 \cdot \Omega_{K,1} + \varepsilon^2 \cdot \Omega_{K,2}}{\Omega_0}$$

$$\lambda_M^* \coloneqq \kappa_{z,K} - \kappa_{z,M}$$

Where:

$$\Omega_0 \coloneqq [\gamma_2 - \gamma_1] \cdot [\bar{s}_1\beta_1\varepsilon - 2(1-\bar{c}_1)] \cdot [\bar{s}_2\beta_2\varepsilon - 2(1-\bar{c}_2)]$$

$$\Omega_{V,1} \coloneqq \beta_1\beta_2\bar{s}_1\bar{s}_2 \cdot [\beta_1^2\gamma_2(\gamma_1 - \omega) - \beta_2^2\gamma_1(\gamma_2 - \omega)]$$

$$\Omega_{V,2} \coloneqq \beta_1^3\gamma_2\bar{s}_1 \cdot 2(1-\bar{c}_2)(\omega - \gamma_1) - \beta_2^3\gamma_1\bar{s}_2 \cdot 2(1-\bar{c}_1)(\omega - \gamma_2)$$

$$\Omega_{M,1} \coloneqq \beta_1\beta_2 \cdot [\beta_1\bar{c}_1\gamma_2\bar{s}_2(\omega - \gamma_1) - \beta_2\bar{c}_2\gamma_1\bar{s}_1(\omega - \gamma_1)]$$

$$\Omega_{M,2} \coloneqq \beta_1^2\bar{c}_1\gamma_2 \cdot 2(\bar{c}_2 - 1)(\omega - \gamma_1) - \beta_2^2\bar{c}_2\gamma_1 \cdot 2(\bar{c}_1 - 1)(\omega - \gamma_2)$$
$$+\beta_1\beta_2\bar{s}_1\bar{s}_2\omega \cdot [\gamma_1 - \gamma_2]$$

$$\Omega_{M,3} \coloneqq \beta_1\gamma_2\bar{s}_1 \cdot 2(1-\bar{c}_2)(\omega - \gamma_1) - \beta_2\gamma_1\bar{s}_2 \cdot 2(1-\bar{c}_1)(\omega - \gamma_2)$$

$$\Omega_{K,1} \coloneqq \beta_1\beta_2 \cdot [\beta_1\gamma_2\bar{s}_2(1-\bar{c}_1)(\gamma_1 - \omega) - \beta_2\gamma_1\bar{s}_1(1-\bar{c}_2)(\gamma_2 - \omega)]$$

$$\Omega_{K,2} \coloneqq 2(\bar{c}_1 - 1)(1-\bar{c}_2) \cdot [\beta_1^2\gamma_2(\gamma_1 - \omega) - \beta_2^2\gamma_1(\gamma_2 - \omega)]$$

Where:

$$\beta_1 := \sqrt{\frac{1}{2}\cdot\frac{\alpha+1}{\alpha-\omega^2} - \sqrt{\frac{1}{4}\cdot\left(\frac{\alpha+1}{\alpha-\omega^2}\right)^2 - \frac{1}{\alpha-\omega^2}}}$$

$$\beta_2 := \sqrt{\frac{1}{2}\cdot\frac{\alpha+1}{\alpha-\omega^2} + \sqrt{\frac{1}{4}\cdot\left(\frac{\alpha+1}{\alpha-\omega^2}\right)^2 - \frac{1}{\alpha-\omega^2}}}$$

$$\gamma_1 := \frac{2\omega}{1-\alpha+\sqrt{\alpha^2-2\alpha+\omega^2+1}}$$

$$\gamma_2 := \frac{2\omega}{1-\alpha-\sqrt{\alpha^2-2\alpha+\omega^2+1}}$$

$$\bar{c}_1 := \cos(\beta_1\varepsilon)$$

$$\bar{c}_2 := \cos(\beta_2\varepsilon)$$

$$\bar{s}_1 := \sin(\beta_1\varepsilon)$$

$$\bar{s}_2 := \sin(\beta_2\varepsilon)$$

Attachment D

The Second Order amplitude functions μ^{II} from 4.3.4 are defined as follows.

$$\mu_{w_A}^{II} := \frac{\Gamma_{wA,1} \cdot \varepsilon^3 + \Gamma_{wA,2} \cdot \varepsilon^2}{\Gamma_0}$$

$$\mu_{\varphi_A}^{II} := \frac{\Gamma_{\varphi A,1} \cdot \varepsilon^3 + \Gamma_{\varphi A,2} \cdot \varepsilon^2 + \Gamma_{\varphi A,3} \cdot \varepsilon}{\Gamma_0}$$

$$\mu_{w_E}^{II} := \frac{\Gamma_{wE,1} \cdot \varepsilon^3 + \Gamma_{wE,2} \cdot \varepsilon^2}{\Gamma_0}$$

$$\mu_{\varphi_E}^{II} := \frac{\Gamma_{\varphi E,1} \cdot \varepsilon^3 + \Gamma_{\varphi E,2} \cdot \varepsilon^2 + \Gamma_{\varphi E,3} \cdot \varepsilon}{\Gamma_0}$$

$$\mu_{v_A}^{II} := \frac{\Gamma_{vA,1} \cdot \varepsilon^3 + \Gamma_{vA,2} \cdot \varepsilon^2}{\Gamma_0}$$

$$\mu_{\psi_A}^{II} := \frac{\Gamma_{\psi A,1} \cdot \varepsilon^3 + \Gamma_{\psi A,2} \cdot \varepsilon^2 + \Gamma_{\psi A,3} \cdot \varepsilon}{\Gamma_0}$$

$$\mu_{v_E}^{II} := \frac{\Gamma_{vE,1} \cdot \varepsilon^3 + \Gamma_{vE,2} \cdot \varepsilon^2}{\Gamma_0}$$

$$\mu_{\psi_E}^{II} := \frac{\Gamma_{\psi E,1} \cdot \varepsilon^3 + \Gamma_{\psi E,2} \cdot \varepsilon^2 + \Gamma_{\psi E,3} \cdot \varepsilon}{\Gamma_0}$$

$$\mu_{wv_A}^{II} := \frac{\Gamma_{wvA,1} \cdot \varepsilon^3 + \Gamma_{wvA,2} \cdot \varepsilon^2}{\Gamma_0}$$

$$\mu_{\varphi\psi_A}^{II} := \frac{\Gamma_{\varphi\psi A,1} \cdot \varepsilon^3 + \Gamma_{\varphi\psi A,2} \cdot \varepsilon^2 + \Gamma_{\varphi\psi A,3} \cdot \varepsilon}{\Gamma_0}$$

$$\mu_{wv_E}^{II} := \frac{\Gamma_{wvE,1} \cdot \varepsilon^3 + \Gamma_{wvE,2} \cdot \varepsilon^2}{\Gamma_0}$$

$$\mu_{\varphi\psi_E}^{II} := \frac{\Gamma_{\varphi\psi E,1} \cdot \varepsilon^3 + \Gamma_{\varphi\psi E,2} \cdot \varepsilon^2 + \Gamma_{\varphi\psi E,3} \cdot \varepsilon}{\Gamma_0}$$

Where:

$$\Gamma_0 := [\gamma_1 - \gamma_2] \cdot [\bar{s}_1 \beta_1 \varepsilon - 2(1 - \bar{c}_1)] \cdot [\bar{s}_2 \beta_2 \varepsilon - 2(1 - \bar{c}_2)]$$

$$\Gamma_{wA,1} \coloneqq \beta_1^2 \beta_2 \bar{s}_2 [\gamma_1 - \omega][\bar{C}_1(\bar{c}_1 - 1) + \bar{S}_1 \bar{s}_1] - \beta_2^2 \beta_1 \bar{s}_1 [\gamma_2 - \omega][\bar{C}_2(\bar{c}_2 - 1) + \bar{S}_2 \bar{s}_2]$$

$$\begin{aligned}\Gamma_{wA,2} \coloneqq\ & 2\beta_1^2[\gamma_1 - \omega][\bar{c}_2 - 1][\bar{C}_1(\bar{c}_1 - 1) + \bar{S}_1 \bar{s}_1] \\ & - 2\beta_2^2[\gamma_2 - \omega][\bar{c}_1 - 1][\bar{C}_2(\bar{c}_2 - 1) + \bar{S}_2 \bar{s}_2]\end{aligned}$$

$$\Gamma_{\varphi A,1} \coloneqq \beta_1^2\beta_2\bar{s}_2[\bar{C}_1\bar{c}_1 + \bar{S}_1\bar{s}_1][\omega - \gamma_1] - \beta_2^2\beta_1\bar{s}_1[\bar{C}_2\bar{c}_2 + \bar{S}_2\bar{s}_2][\omega - \gamma_2]$$

$$\begin{aligned}\Gamma_{\varphi A,2} \coloneqq\ & \bar{C}_1[\gamma_1 - \omega][\beta_1^2\bar{c}_1 \cdot 2(1 - \bar{c}_2) + \beta_1\beta_2\bar{s}_1\bar{s}_2]\\ & + \bar{S}_1[\gamma_1 - \omega][\beta_1^2\bar{s}_1 \cdot 2(1 - \bar{c}_2) + \beta_1\beta_2\bar{s}_1(1 - \bar{c}_1)]\\ & - \bar{C}_2[\gamma_2 - \omega][\beta_2^2\bar{c}_2 \cdot 2(1 - \bar{c}_1) + \beta_1\beta_2\bar{s}_1\bar{s}_2]\\ & - \bar{S}_2[\gamma_2 - \omega][\beta_2^2\bar{s}_2 \cdot 2(1 - \bar{c}_1) + \beta_1\beta_2\bar{s}_2(1 - \bar{c}_2)]\end{aligned}$$

$$\begin{aligned}\Gamma_{\varphi A,3} \coloneqq\ & \beta_1[\bar{C}_1\bar{s}_1 - \bar{S}_1(\bar{c}_1 - 1)][2 - 2\bar{c}_2][\omega - \gamma_1]\\ & - \beta_2[\bar{C}_2\bar{s}_2 - \bar{S}_2(\bar{c}_2 - 1)][2 - 2\bar{c}_1][\omega - \gamma_2]\end{aligned}$$

$$\Gamma_{wE,1} \coloneqq \beta_1^2\beta_2\bar{s}_2[\bar{S}_1\bar{s}_1 + \bar{C}_1(\bar{c}_1 - 1)][\omega - \gamma_1] - \beta_2^2\beta_1\bar{s}_1[\bar{S}_2\bar{s}_2 + \bar{C}_2(\bar{c}_2 - 1)][\omega - \gamma_2]$$

$$\begin{aligned}\Gamma_{wE,2} \coloneqq\ & \beta_1^2[\bar{S}_1\bar{s}_1 + \bar{C}_1(\bar{c}_1 - 1)][2 - 2\bar{c}_2][\gamma_1 - \omega]\\ & - \beta_2^2[\bar{S}_2\bar{s}_2 + \bar{C}_2(\bar{c}_2 - 1)][2 - 2\bar{c}_1][\gamma_2 - \omega]\end{aligned}$$

$$\Gamma_{\varphi E,1} \coloneqq \beta_1\beta_2[\bar{C}_1\beta_1\bar{s}_2(\gamma_1 - \omega) - \bar{C}_2\beta_2\bar{s}_1(\gamma_2 - \omega)]$$

$$\begin{aligned}\Gamma_{\varphi E,2} \coloneqq\ & \beta_1[\omega - \gamma_1][\bar{C}_1(2\beta_1 - 2\beta_1\bar{c}_2 + \beta_2\bar{s}_1\bar{s}_2) + \bar{S}_1\beta_2\bar{s}_2(1 - \bar{c}_1)]\\ & - \beta_2[\omega - \gamma_2][\bar{C}_2(2\beta_2 - 2\beta_2\bar{c}_1 + \beta_1\bar{s}_1\bar{s}_2) + \bar{S}_2\beta_1\bar{s}_1(1 - \bar{c}_2)]\end{aligned}$$

$$\begin{aligned}\Gamma_{\varphi E,3} \coloneqq\ & \beta_1[\bar{C}_1\bar{s}_1 - \bar{S}_1(\bar{c}_1 - 1)][2 - 2\bar{c}_2][\gamma_1 - \omega]\\ & - \beta_2[\bar{C}_2\bar{s}_2 - \bar{S}_2(\bar{c}_2 - 1)][2 - 2\bar{c}_1][\gamma_2 - \omega]\end{aligned}$$

$$\begin{aligned}\Gamma_{vA,1} \coloneqq\ & -\beta_1^2\beta_2\bar{s}_2\gamma_2[\bar{S}_1\bar{s}_1 + \bar{C}_1(\bar{c}_1 - 1)][\gamma_1\omega - \alpha]\\ & + \beta_2^2\beta_1\bar{s}_1\gamma_1[\bar{S}_2\bar{s}_2 + \bar{C}_2(\bar{c}_2 - 1)][\gamma_2\omega - \alpha]\end{aligned}$$

$$\begin{aligned}\Gamma_{vA,2} \coloneqq\ & \beta_1^2\gamma_2[\bar{S}_1\bar{s}_1 + \bar{C}_1(\bar{c}_1 - 1)][2 - 2\bar{c}_2][\gamma_1\omega - \alpha]\\ & - \beta_2^2\gamma_1[\bar{S}_2\bar{s}_2 + \bar{C}_2(\bar{c}_2 - 1)][2 - 2\bar{c}_1][\gamma_2\omega - \alpha]\end{aligned}$$

$$\Gamma_{\psi A,1} \coloneqq -\beta_1^2\beta_2\gamma_2\bar{s}_2[\bar{S}_1\bar{s}_1 + \bar{C}_1\bar{c}_1][\gamma_1\omega - \alpha] + \beta_2^2\beta_1\gamma_1\bar{s}_1[\bar{S}_2\bar{s}_2 + \bar{C}_2\bar{c}_2][\gamma_2\omega - \alpha]$$

$$\begin{aligned}\Gamma_{\psi A,2} \coloneqq\ & \beta_1\gamma_2[\beta_1(\bar{S}_1\bar{s}_1 + \bar{C}_1\bar{c}_1)(2 - 2\bar{c}_2) + \beta_2\bar{s}_2(\bar{C}_1\bar{s}_1 - \bar{S}_1\bar{c}_1 + \bar{S}_1)][\gamma_1\omega - \alpha]\\ & - \beta_2\gamma_1[\beta_2(\bar{S}_2\bar{s}_2 + \bar{C}_2\bar{c}_2)(2 - 2\bar{c}_1) + \beta_1\bar{s}_1(\bar{C}_2\bar{s}_2 - \bar{S}_2\bar{c}_2 + \bar{S}_2)][\gamma_2\omega - \alpha]\end{aligned}$$

$$\begin{aligned}\Gamma_{\psi A,3} \coloneqq\ & \beta_1\gamma_2[\bar{S}_1(\bar{c}_1 - 1) - \bar{C}_1\bar{s}_1][2 - 2\bar{c}_2][\gamma_1\omega - \alpha]\\ & - \beta_2\gamma_1[\bar{S}_2(\bar{c}_2 - 1) - \bar{C}_2\bar{s}_2][2 - 2\bar{c}_1][\gamma_2\omega - \alpha]\end{aligned}$$

$$\begin{aligned}\Gamma_{vE,1} \coloneqq\ & \beta_1^2\beta_2\gamma_2\bar{s}_2[\bar{S}_1\bar{s}_1 + \bar{C}_1(\bar{c}_1 - 1)][\gamma_1\omega - \alpha]\\ & - \beta_2^2\beta_1\gamma_1\bar{s}_1[\bar{S}_2\bar{s}_2 + \bar{C}_2(\bar{c}_2 - 1)][\gamma_2\omega - \alpha]\end{aligned}$$

$$\begin{aligned}\Gamma_{vE,2} \coloneqq\ & -\beta_1^2\gamma_2[\bar{C}_1(\bar{c}_1 - 1) + \bar{S}_1\bar{s}_1][2 - 2\bar{c}_2][\gamma_1\omega - \alpha] -\\ & \coloneqq -\beta_2^2\gamma_1[\bar{C}_2(\bar{c}_2 - 1) + \bar{S}_2\bar{s}_2][2 - 2\bar{c}_1][\gamma_2\omega - \alpha]\end{aligned}$$

$$
\begin{aligned}
\Gamma_{\psi E,1} &:= \bar{C}_1\beta_1^2\beta_2\gamma_2\bar{s}_2[\gamma_1\omega-\alpha]-\bar{C}_2\beta_2^2\beta_1\gamma_1\bar{s}_1[\gamma_2\omega-\alpha]\\
\Gamma_{\psi E,2} &:= -\bar{C}_1\beta_1\gamma_2[2\beta_1(1-\bar{c}_2)+\beta_2\bar{s}_1\bar{s}_2][\gamma_1\omega-\alpha]+\bar{S}_1\beta_1\beta_2\gamma_2\bar{s}_2[\bar{c}_1-1][\gamma_1\omega-\alpha]\\
&\qquad +\bar{C}_2\beta_2\gamma_1[2\beta_2(1-\bar{c}_1)+\beta_1\bar{s}_1\bar{s}_2][\gamma_2\omega-\alpha]\\
&\qquad -\bar{S}_2\beta_2\beta_1\gamma_1\bar{s}_1[\bar{c}_2-1][\gamma_2\omega-\alpha]\\
\Gamma_{\psi E,3} &:= \beta_1\gamma_2[\bar{C}_1\bar{s}_1-\bar{S}_1(\bar{c}_1-1)][2-2\bar{c}_2][\gamma_1\omega-\alpha]\\
&\qquad -\beta_2\gamma_1[\bar{C}_2\bar{s}_2-\bar{S}_2(\bar{c}_2-1)][2-2\bar{c}_1][\gamma_2\omega-\alpha]\\
\Gamma_{wvA,1} &:= \beta_1^2\beta_2\gamma_2\bar{s}_2[\bar{S}_1\bar{s}_1+\bar{C}_1(\bar{c}_1-1)][\omega-\gamma_1]-\beta_2^2\beta_1\gamma_1\bar{s}_1[\bar{S}_2\bar{s}_2+\bar{C}_2(\bar{c}_2-1)][\omega-\gamma_2]\\
\Gamma_{wvA,2} &:= -\beta_1^2\gamma_2[\bar{S}_1\bar{s}_1+\bar{C}_1(\bar{c}_1-1)][2-2\bar{c}_2][\omega-\gamma_1]\\
&\qquad +\beta_2^2\gamma_1[\bar{S}_2\bar{s}_2+\bar{C}_2(\bar{c}_2-1)][2-2\bar{c}_1][\omega-\gamma_2]\\
\Gamma_{\varphi\psi A,1} &:= \beta_1^2\beta_2\gamma_2\bar{s}_2[\bar{S}_1\bar{s}_1+\bar{C}_1\bar{c}_1][\omega-\gamma_1]-\beta_2^2\beta_1\gamma_1\bar{s}_1[\bar{S}_2\bar{s}_2+\bar{C}_2\bar{c}_2][\omega-\gamma_2]\\
\Gamma_{\varphi\psi A,2} &:= -\bar{C}_1\beta_1\gamma_2[2\beta_1\bar{c}_1(1-\bar{c}_2)+\beta_2\bar{s}_1\bar{s}_2][\omega-\gamma_1]\\
&\qquad -\bar{S}_1\beta_1\gamma_2[2\beta_1\bar{s}_1(1-\bar{c}_2)+\beta_2\bar{s}_2(1-\bar{c}_1)][\omega-\gamma_1]\\
&\qquad +\bar{C}_2\beta_2\gamma_1[2\beta_2\bar{c}_2(1-\bar{c}_1)+\beta_1\bar{s}_1\bar{s}_2][\omega-\gamma_2]\\
&\qquad +\bar{S}_2\beta_2\gamma_1[2\beta_2\bar{s}_2(1-\bar{c}_1)+\beta_1\bar{s}_1(1-\bar{c}_2)][\omega-\gamma_2]\\
\Gamma_{\varphi\psi A,3} &:= \beta_1\gamma_2[\bar{C}_1\bar{s}_1-\bar{S}_1(\bar{c}_1-1)][2-2\bar{c}_2][\omega-\gamma_1]\\
&\qquad -\beta_2\gamma_1[\bar{C}_2\bar{s}_2-\bar{S}_2(\bar{c}_2-1)][2-2\bar{c}_1][\omega-\gamma_2]\\
\Gamma_{wvE,1} &:= -\beta_1^2\beta_2\gamma_2\bar{s}_2[\bar{S}_1\bar{s}_1+\bar{C}_1(\bar{c}_1-1)][\omega-\gamma_1]\\
&\qquad +\beta_2^2\beta_1\gamma_1\bar{s}_1[\bar{S}_2\bar{s}_2+\bar{C}_2(\bar{c}_2-1)][\omega-\gamma_2]\\
\Gamma_{wvE,2} &:= \beta_1^2\gamma_2[\bar{S}_1\bar{s}_1+\bar{C}_1(\bar{c}_1-1)][2-2\bar{c}_2][\omega-\gamma_1]\\
&\qquad -\beta_2^2\gamma_1[\bar{S}_2\bar{s}_2+\bar{C}_2(\bar{c}_2-1)][2-2\bar{c}_1][\omega-\gamma_2]\\
\Gamma_{\varphi\psi E,1} &:= -\bar{C}_1\beta_1^2\beta_2\gamma_2\bar{s}_2[\omega-\gamma_1]+\bar{C}_2\beta_2^2\beta_1\gamma_1\bar{s}_1[\omega-\gamma_2]\\
\Gamma_{\varphi\psi E,2} &:= \beta_1\gamma_2\big[\bar{C}_1[2\beta_1(1-\bar{c}_2)+\beta_2\bar{s}_1\bar{s}_2]+\bar{S}_1\beta_2\bar{s}_2[1-\bar{c}_1]\big][\omega-\gamma_1]\\
&\qquad -\beta_2\gamma_1\big[\bar{C}_2[2\beta_2(1-\bar{c}_1)+\beta_1\bar{s}_1\bar{s}_2]+\bar{S}_2\beta_1\bar{s}_1[1-\bar{c}_2]\big][\omega-\gamma_2]\\
\Gamma_{\varphi\psi E,3} &:= -\beta_1\gamma_2[\bar{C}_1\bar{s}_1-\bar{S}_1(\bar{c}_1-1)][2-2\bar{c}_2][\omega-\gamma_1]\\
&\qquad +\beta_2\gamma_1[\bar{C}_2\bar{s}_2-\bar{S}_2(\bar{c}_2-1)][2-2\bar{c}_1][\omega-\gamma_2]
\end{aligned}
$$

Where:

$$\beta_1 := \sqrt{\frac{1}{2}\cdot\frac{\alpha+1}{\alpha-\omega^2} - \sqrt{\frac{1}{4}\cdot\left(\frac{\alpha+1}{\alpha-\omega^2}\right)^2 - \frac{1}{\alpha-\omega^2}}}$$

$$\beta_2 := \sqrt{\frac{1}{2}\cdot\frac{\alpha+1}{\alpha-\omega^2} + \sqrt{\frac{1}{4}\cdot\left(\frac{\alpha+1}{\alpha-\omega^2}\right)^2 - \frac{1}{\alpha-\omega^2}}}$$

$$\gamma_1 := \frac{2\omega}{1-\alpha+\sqrt{\alpha^2-2\alpha+\omega^2+1}}$$

$$\gamma_2 := \frac{2\omega}{1-\alpha-\sqrt{\alpha^2-2\alpha+\omega^2+1}}$$

$$\bar{c}_1 := \cos(\beta_1\varepsilon)$$

$$\bar{c}_2 := \cos(\beta_2\varepsilon)$$

$$\bar{s}_1 := \sin(\beta_1\varepsilon)$$

$$\bar{s}_2 := \sin(\beta_2\varepsilon)$$

$$\bar{C}_1 := \cos(\beta_1\varepsilon\cdot\xi)$$

$$\bar{C}_2 := \cos(\beta_2\varepsilon\cdot\xi)$$

$$\bar{S}_1 := \sin(\beta_1\varepsilon\cdot\xi)$$

$$\bar{S}_2 := \sin(\beta_2\varepsilon\cdot\xi)$$

Attachment E

The derived stiffness coefficients and amplitude functions have been inputted in the graphing calculator software “desmos”. The software allows the user to interact with the functions by changing the initial parameters. Apart from the showing the amplitude functions of First Order Analysis, and the stiffness coefficients and bending moment amplitude functions from Second Order Analysis, the file includes the exemplary analysis of the Free-Fixed Column from 4.4.2.

Link:
https://www.desmos.com/calculator/m1v93wrggu