0074 – Frame Structure Subjected to Earthquake Loading

Description

A two-storey single-bay frame structure is subjected to earthquake loading. Modulus of elasticity and cross-section of the frame beams are much larger than those of the columns so the beams can be considered rigid.

Neglecting self-weight and assuming the lumped masses are at the floor levels, determine natural frequencies of the structure. For each obtained frequency specify the standardized displacements of the floors as well as equivalent forces generated using the elastic response spectrum according to the standard SIA 261/1:2003.

Material	Columns	Modulus of Elasticity	E_c	48.000	GPa
		Poisson's Ratio	ν_c	0.500	–
	Beams	Modulus of Elasticity	E_b	1×10^{10}	GPa
		Poisson's Ratio	ν_b	0.500	–
Geometry	Columns	Width	w_c	0.500	m
		Height	h	0.500	m
		Length	l	5.000	m
	Beams	Width	w_b	4.000	m
		Height	h	0.500	m
		Length	l	5.000	m
Mass	Floor	First	m_1	5×10^5	kg
		Second	m_2	5×10^5	kg

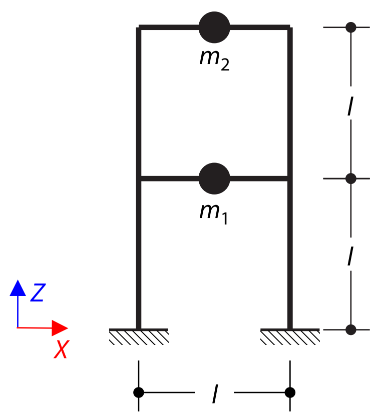

Figure 1: Problem sketch

Analytical Solution

Free Vibration

Due to the rigid beams and lumped masses, the structure can be idealized as a shear building with two degrees of freedom. Its undamped free vibration is described by the differential equation

$$\boldsymbol{M}\ddot{\boldsymbol{u}} + \boldsymbol{K}\boldsymbol{u} = 0 \tag{74 – 1}$$

where $\boldsymbol{u}$ is the displacement vector, $\boldsymbol{M}$ is the mass matrix

$$\boldsymbol{M} = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \tag{74 – 2}$$

and $\boldsymbol{K}$ is the stiffness matrix

$$\boldsymbol{K} = \begin{bmatrix} 4k & -2k \\ -2k & 2k \end{bmatrix} \tag{74 – 3}$$

where k is the flexural stiffness of a column in the X-direction which is defined according to the direct stiffness method as reaction force R_x developed on a beam with both ends fixed by displacing one of its supports by unit deformation $d_x = 1$

$$k = \frac{R_x}{d_x} = \frac{R_x}{1} = \frac{12EI}{l^3} = \frac{12Ew_c h^3}{12l^3} = \frac{Ew_c h^3}{l^3} \tag{74 – 4}$$

The displacement is assumed to be harmonic in time, that is

$$\boldsymbol{u}(t) = \boldsymbol{U}e^{i\omega t} \tag{74 – 5}$$

where ω is the angular frequency of the system and $\boldsymbol{U}$ the corresponding eigenvector, hence, equation **(74 – 1)** becomes

$$(\boldsymbol{K} - \omega^2\boldsymbol{M})\boldsymbol{U} = 0 \tag{74 – 6}$$

Finding the non-trivial solution $\boldsymbol{U} \neq 0$ of **(74 – 6)** leads then to the requirement that the determinant of the matrix must be zero, more precisely

$$|\boldsymbol{K} - \omega^2\boldsymbol{M}| = 0 \tag{74 – 7}$$

which results in a second-order polynomial in terms of ω, the roots of which are called eigenvalues. These eigenvalue angular frequencies ω_i can easily be transformed to the natural frequencies $f_i = \omega_i/2\pi$ of the structure

$$f_1 = \frac{\omega_1}{2\pi} \approx 0.964 \text{ Hz} \quad (74-8)$$

$$f_2 = \frac{\omega_2}{2\pi} \approx 2.523 \text{ Hz} \quad (74-9)$$

For each frequency f_i, an eigenvector $\boldsymbol{U}_i$ satisfying equation **(74 – 6)** is determined. However, because the matrix $(\boldsymbol{K} - \omega^2 \boldsymbol{M})$ is singular, $\boldsymbol{U}$ is not uniquely determined and only standardized displacements can be obtained. Setting $U_{1,2} = 1$ and $U_{2,1} = 1$, the eigenvectors $\boldsymbol{U}_1$ and $\boldsymbol{U}_2$ are

$$\boldsymbol{U}_1 = \begin{bmatrix} U_{1,1} \\ U_{1,2} \end{bmatrix} = \begin{bmatrix} 0.618 \\ 1.000 \end{bmatrix} \quad (74-10)$$

$$\boldsymbol{U}_2 = \begin{bmatrix} U_{2,1} \\ U_{2,2} \end{bmatrix} = \begin{bmatrix} 1.000 \\ -0.618 \end{bmatrix} \quad (74-11)$$

Forced Vibration

The displacement $\boldsymbol{u}$ can be written as a linear combination of the eigenvectors $\boldsymbol{U}_i$

$$\boldsymbol{u} = \sum_{i=1}^{2} \xi_i \boldsymbol{U}_i \quad (74-12)$$

for some real constants ξ_i, and the forced undamped vibration then reads as

$$\boldsymbol{M} \sum_{i=1}^{2} \ddot{\xi}_i \boldsymbol{U}_i + \boldsymbol{K} \sum_{i=1}^{2} \xi_i \boldsymbol{U}_i = \boldsymbol{M}\boldsymbol{A} \ddot{u}_g(t) \quad (74-13)$$

where $\boldsymbol{A} = [1, 1]$ and $\ddot{u}_g(t)$ is the ground acceleration given as a function of time. Now, multiplying **(74 – 13)** by the eigenvector $\boldsymbol{U}_j$

$$\boldsymbol{U}_j^T \boldsymbol{M} \sum_{i=1}^{2} \ddot{\xi}_i \boldsymbol{U}_i + \boldsymbol{U}_j^T \boldsymbol{K} \sum_{i=1}^{2} \xi_i \boldsymbol{U}_i = \boldsymbol{U}_j^T \boldsymbol{M}\boldsymbol{A} \ddot{u}_g(t) \quad (74-14)$$

when the eigenvectors $\boldsymbol{U}_i$ are orthogonal, that is $\boldsymbol{U}_j^T \boldsymbol{M} \boldsymbol{U}_i = 0$ for $i \neq j$, and from **(74 – 6)** there is $\boldsymbol{K} = \omega^2 \boldsymbol{M}$, at last

$$\ddot{\xi}_j + \omega^2 \xi_j = \alpha_j \ddot{u}_g(t) \quad (74-15)$$

where α_j is the participation factor

$$\alpha_j = \frac{\boldsymbol{U}_j^T \boldsymbol{M}\boldsymbol{A}}{\boldsymbol{U}_j^T \boldsymbol{M} \boldsymbol{U}_j} \quad (74-16)$$

By multiplying the displacement, velocity and acceleration of the system by the participation factor their maximum values can be found. Therefore Newton's second law of motion for the equivalent forces generated at each floor takes the following form

$$\boldsymbol{F}_i = \boldsymbol{M}\boldsymbol{U}_i \alpha_i S_{a_i} \tag{74 – 17}$$

where the acceleration S_{a_i} for each mode is obtained from the given design spectrum (**Figure 2**)

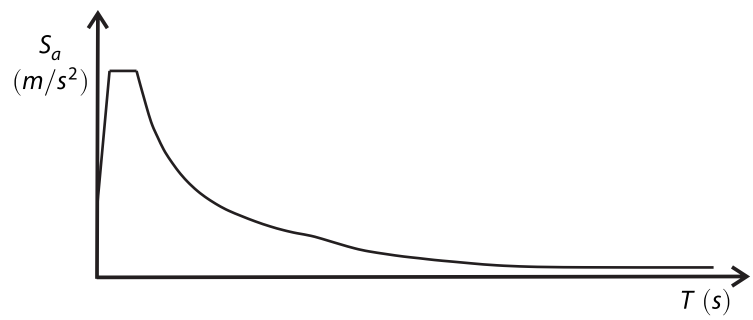

Figure 2: Elastic Response Spectrum SIA 261/1:2003

$$T_1 = \frac{1}{f_1} \approx 1.038 \text{ s} \quad \Rightarrow \quad S_{a_1} \approx 0.5782 \text{ m/s}^2 \tag{74 – 18}$$

$$T_2 = \frac{1}{f_2} \approx 0.396 \text{ s} \quad \Rightarrow \quad S_{a_2} \approx 1.5000 \text{ m/s}^2 \tag{74 – 19}$$

hence

$$\boldsymbol{F}_1 = \begin{bmatrix} F_{1,1} \\ F_{1,2} \end{bmatrix} \approx \begin{bmatrix} 209.195 \\ 338.481 \end{bmatrix} \text{kN} \tag{74 – 20}$$

$$\boldsymbol{F}_2 = \begin{bmatrix} F_{2,1} \\ F_{2,2} \end{bmatrix} \approx \begin{bmatrix} -207.295 \\ 128.115 \end{bmatrix} \text{kN} \tag{74 – 21}$$

RFEM 5 and RSTAB 8 Settings

- Modeled in version RFEM 5.07.03 and RSTAB 8.07.03
- Geometrically linear analysis is considered
- Mass is considered in the *X*-direction
- Diagonal mass matrix is generated
- Root of characteristic polynomial is used as solving method

Results

Structure File	Program
0074.01	RF-DYNAM Pro
0074.02	DYNAM Pro

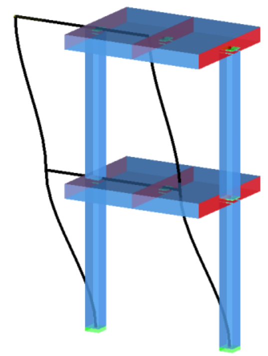

Figure 3: Deformation in the first mode

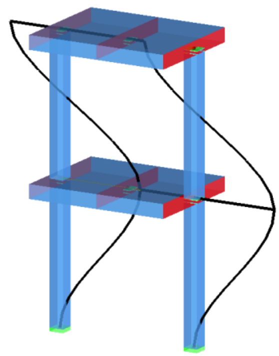

Figure 4: Deformation in the second mode

As can be seen from the tables below, good agreement of the analytical result with numerical output was achieved.

Natural Frequencies

Mode	Analytical Solution	RF-DYNAM Pro		DYNAM Pro	
	f_n [Hz]	f_n [Hz]	Ratio [-]	f_n [Hz]	Ratio [-]
First	0.964	0.964	1.000	0.964	1.000
Second	2.523	2.523	1.000	2.523	1.000

Standardized Displacement of the First Floor

Mode	Analytical Solution	RF-DYNAM Pro		DYNAM Pro	
	$U_{n,1}$ [-]	$U_{n,1}$ [-]	Ratio [-]	$U_{n,1}$ [-]	Ratio [-]
First	0.618	0.618	1.000	0.618	1.000
Second	1.000	1.000	1.000	1.000	1.000

Standardized Displacement of the Second Floor

Mode	Analytical Solution	RF-DYNAM Pro		DYNAM Pro	
	$U_{n,2}$ [-]	$U_{n,2}$ [-]	Ratio [-]	$U_{n,2}$ [-]	Ratio [-]
First	1.000	1.000	1.000	1.000	1.000
Second	-0.618	-0.618	1.000	-0.618	1.000

Equivalent Force Generated at the First Floor

Mode	Analytical Solution	RF-DYNAM Pro		DYNAM Pro	
	$F_{n,1}$ [-]	$F_{n,1}$ [-]	Ratio [-]	$F_{n,1}$ [-]	Ratio [-]
First	209.195	209.188	1.000	209.203	1.000
Second	-207.295	-207.285	1.000	-207.293	1.000

Equivalent Force Generated at the Second Floor

Mode	Analytical Solution	RF-DYNAM Pro		DYNAM Pro	
	$F_{n,2}$ [-]	$F_{n,2}$ [-]	Ratio [-]	$F_{n,2}$ [-]	Ratio [-]
First	338.484	338.474	1.000	338.497	1.000
Second	128.115	128.112	1.000	128.115	1.000