0079 – Catenary

Description

A very stiff cable of length L and specific mass μ is suspended between two supports at distance d, in accord with **Figure 1**. Determine the equilibrium shape of the cable, the so-called catenary, consider the gravitational acceleration g and neglect the stiffness of the cable. Verify the position of the cable at given test points. The problem is described by the following set of parameters.

Material	Steel Cable	Modulus of Elasticity	E	210000.000	MPa
		Poisson's Ratio	ν	0.300	–
		Specific Mass	μ	2.466	kg/m
Geometry		Supports Distance	d	5.000	m
		Cable Length	L	5.036	m
Load		Gravitational Acceleration	g	9.810	ms^{-2}

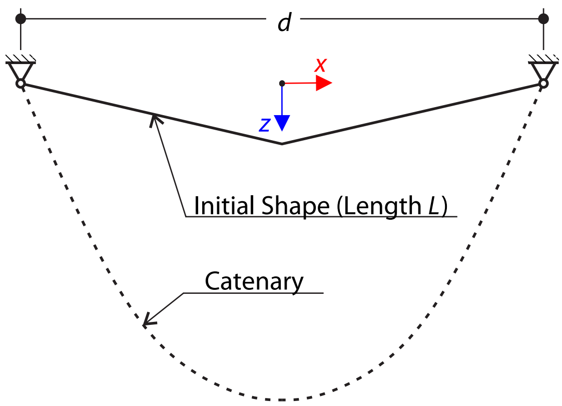

Figure 1: Problem Sketch

Analytical Solution

The final solution is a catenary where each point of the cable has minimal potential energy, namely, the solution of a certain minimization problem. The potential energy of the cable reads as follows

$$E_p(z) = \int_{\mathcal{L}(z)} gz\mu \, \mathrm{d}s \tag{79 – 1}$$

where the cable $\mathcal{L}(z)$ is described by a height function $z(x)\colon \left[-\frac{d}{2}, \frac{d}{2}\right] \to \mathbb{R}$, namely

$$\mathcal{L}(z) = \left\{ \mathbf{s} := (x, z(x)) \in \mathbb{R}^2 \colon |x| \leq \frac{d}{2} \right\}, \tag{79 – 2}$$

while the origin of the coordinate system is set at the middle between the cable supports. In turn, **(79 – 1)** can be expressed as

$$E_p(z) = \int_{-d/2}^{d/2} \mu g z \sqrt{1 + (z')^2} \, \mathrm{d}x =: \int_{-d/2}^{d/2} F(z(x), z'(x)) \, \mathrm{d}x \tag{79 – 3}$$

Hence, the minimization problem for the total potential energy takes the form

$$\text{Minimize} \quad E_p(z), \tag{79 – 4}$$

$$\text{subject to} \quad z \in C^1 \left(\left[-\frac{d}{2}, \frac{d}{2} \right] \right), \tag{79 – 5}$$

$$z\left(-\frac{d}{2}\right) = 0 = z\left(\frac{d}{2}\right), \tag{79 – 6}$$

$$\int_{\mathcal{L}(z)} \mathrm{d}s = \int_{-d/2}^{d/2} \sqrt{1 + (z')^2} \, \mathrm{d}x = L \tag{79 – 7}$$

where the last constraint states that the total length of the cable is equal to L.

The stationary solution of the variational problem **(79 – 4)**–**(79 – 7)** satisfies the Euler–Lagrange equation **(79 – 8)**,

$$\frac{\partial F}{\partial z} - \frac{\mathrm{d}}{\mathrm{d}x} \frac{\partial F}{\partial z'} = 0, \tag{79 – 8}$$

where F is defined in **(79 – 3)**, however, as F does not depend explicitly on x, the more convenient modification of the Euler–Lagrange equation, the so-called Beltrami identity **(79 – 9)**, has to hold

$$F - z' \frac{\partial F}{\partial z'} = K, \qquad K \in \mathbb{R} \tag{79 – 9}$$

Substituting into **(79 – 9)**, the following first-order differential equation is obtained

$$z = A\sqrt{1 + (z')^2}, \qquad A = \frac{K}{\mu g} \tag{79 – 10}$$

which admits, at the end, the solution

$$z(x) = A \cosh\left(\frac{x + C}{A}\right), \quad C \in \mathbb{R} \tag{79 – 11}$$

The real constants A and C are computed from the length constraint **(79 – 7)** and the boundary condition **(79 – 6)**, respectively. The sought catenary function takes the form of

$$z(x) \approx 12.041 \cosh\left(\frac{x}{12.041}\right) \quad [\text{m}] \qquad (79-12)$$

Please note that the function **(79 – 12)** is further shifted, so that the support nodes have the *z*-coordinate equal to zero, see **Figure 2** and **(79 – 6)**. Similarly, the RFEM 5 / RSTAB 8 results (deflections) have to be post-processed for the comparison with the analytical result.

RFEM 5 and RSTAB 8 Settings

- Modeled in RFEM 5.09.01 and RFEM 8.09.01
- The cable is divided into 100 parts
- The number of increments is 10
- Isotropic linear elastic model is used

Results

Structure Files	Program
0079.01	RFEM 5
0079.02	RSTAB 8

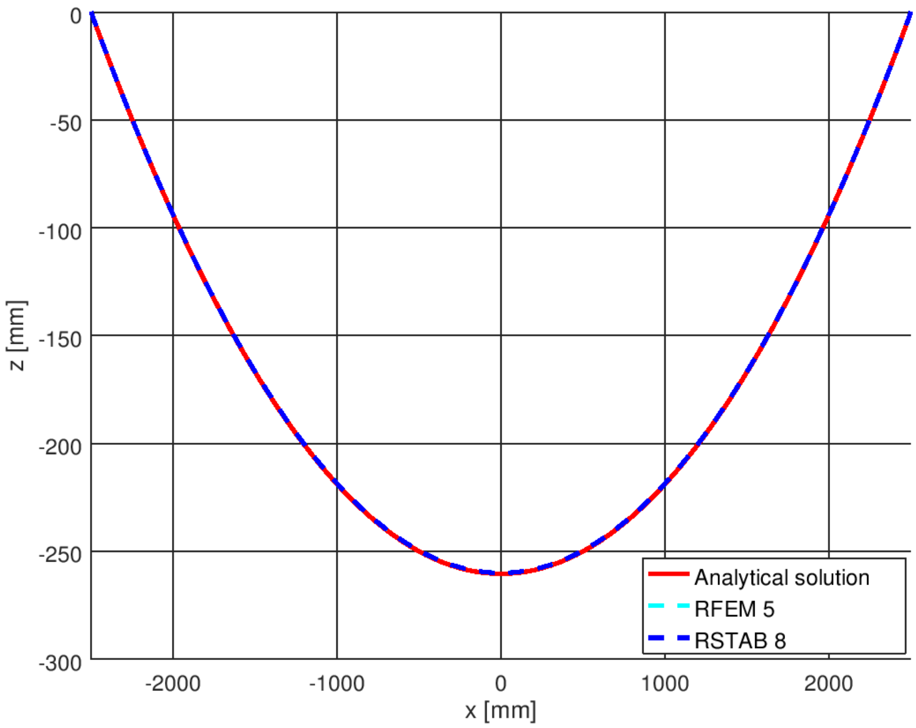

Figure 2: Comparison of the analytical results and RFEM 5 / RSTAB 8 results

Height [mm]	Distance x [mm]	Analytical solution	RFEM 5	Ratio [-]	RSTAB 8	Ratio [-]
z	0	-260.461	-260.078	0.999	-260.088	0.999
	1006.012	-218.412	-218.090	0.999	-218.098	0.999

Remark: The second test point $x = 1006.012$ mm is chosen from the final position of the test node.