0106 – Natural Vibrations of a String

Description

A thin string of diameter D is tensioned by force N according to **Figure 1**. Determine the natural frequencies of the string. The problem is described by the following parameters.

Material	Steel	Modulus of Elasticity	E	210000.0	MPa
		Poisson's Ratio	ν	0.300	–
		Linear Density	μ	24.662	kgm^{-1}
Geometry		Diameter	D	0.002	m
		Length	L	1.000	m
Load		Tension Force	N	1000.000	N

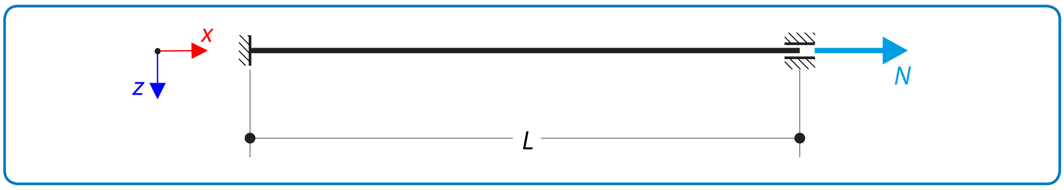

Figure 1: Problem Sketch

Analytical Solution

Free vibrations of a string are described by the wave equation in the following form

$$\frac{\partial^2 u_z}{\partial x^2}(x,t) - \frac{1}{c^2}\frac{\partial^2 u_z}{\partial t^2}(x,t) = 0 \qquad (106-1)$$

The speed of the wave propagation c is given by the linear density of the string μ and the tension force N

$$c = \sqrt{\frac{N}{\mu}} \qquad (106-2)$$

The solution is sought for through separation of variables

$$u_z(x,t) = X(x)T(t) \qquad (106-3)$$

Substituting into **(106 – 1)** yields[1]

[1] The dashed notation indicates the derivative with respect to the space coordinate $X'' = \frac{d^2X(x)}{dx^2}$, while the dotted notation with respect to time t.

$$\frac{\ddot{T}}{T} = c^2 \frac{X''}{X} = -\Omega^2 \tag{106 – 4}$$

The left-hand side depends only on time t, while the right-hand side only on the spacial coordinate x. Thus, both sides have to be equal to a constant, which can be shown to be negative, $-\Omega^2$, for some $\Omega > 0$.

The first part of **(106 – 4)**

$$\ddot{T} + \Omega^2 T = 0 \tag{106 – 5}$$

gives a solution in the following form

$$T(t) = A \sin(\Omega t) + B \cos(\Omega t) \tag{106 – 6}$$

where A, B depend on the initial conditions. The solution of the second part of **(106 – 4)**

$$X'' + \frac{\Omega^2}{c^2} X = 0 \tag{106 – 7}$$

looks analogous

$$X(x) = C \sin\left(\frac{\Omega}{c} x\right) + D \cos\left(\frac{\Omega}{c} x\right) \tag{106 – 8}$$

and the constants C, D depend on the boundary conditions, namely, the deflection on both ends is equal to zero

$$X(0) = 0 \tag{106 – 9}$$

$$X(L) = 0 \tag{106 – 10}$$

Using the first boundary condition, there is $D = 0$, on the other hand, the second boundary condition yields

$$C \sin\left(\frac{\Omega}{c} L\right) = 0 \tag{106 – 11}$$

the solution of which is determined by the roots of the sine function, more precisely

$$\Omega_n = c\frac{n\pi}{L}, \qquad n = 1, 2, 3, \ldots \tag{106 – 12}$$

Considering that $\Omega_n = 2\pi f_n$, the natural frequencies of the string can be calculated as

$$f_n = \frac{n}{2L}\sqrt{\frac{N}{\mu}}, \qquad n = 1, 2, 3, \ldots \qquad (106-13)$$

and the general solution of the free vibration of the string is the sum of the mode shapes

$$u_z(x,t) = \sum_n \sin\left(\frac{\Omega_n}{c}x\right)\left(A_n\cos(\Omega_n t) + B_n\sin(\Omega_n t)\right) \qquad (106-14)$$

where A_n and B_n are integration constants determined by the initial conditions.

RFEM 5 and RSTAB 8 Settings

- Modeled in RFEM 5.16.01 and RSTAB 8.16.01
- The string is modeled by a member of type Cable
- The member is divided into 100 parts
- Isotropic linear elastic material model is used
- Shear stiffness of members is deactivated

Results

Structure Files	Program
0106.01	RFEM 5 – RF-DYNAM Pro
0106.02	RSTAB 8 – DYNAM Pro

Quantity	Analytical Solution	RFEM 5	Ratio	RSTAB 8	Ratio
f_1 [Hz]	100.683	100.684	1.000	100.680	1.000
f_2 [Hz]	201.366	201.367	1.000	201.335	1.000
f_3 [Hz]	302.049	302.050	1.000	301.940	1.000
f_4 [Hz]	402.731	402.733	1.000	402.471	0.999

Following **Figure 2** shows the first four natural shapes of the investigated string.

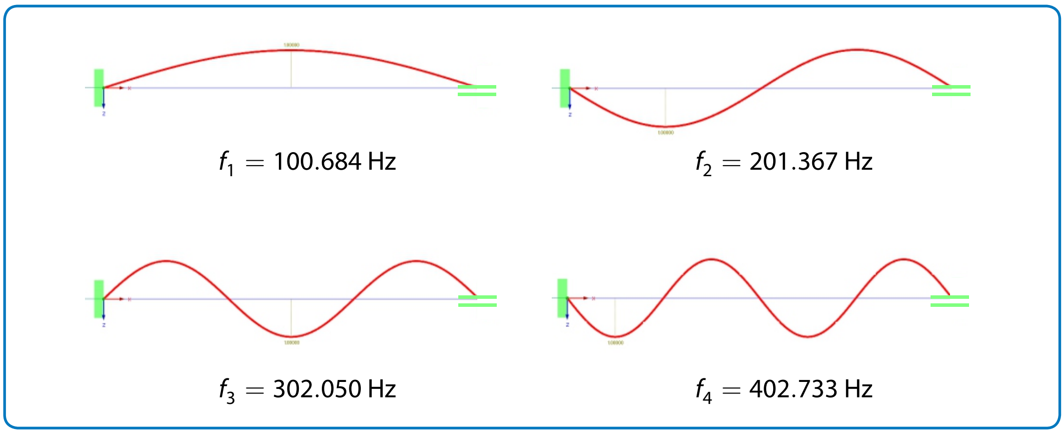

Figure 2: First four natural shapes of the string in RFEM 5