0001 – Torsional Constant and Polar Moment of Inertia

Description

A cross-section of the tube (annular area), shown in the **Figure 1**, is the rotationally symmetrical with respect to the *x*-axis. Determine the torsional constant[1] J for this cross-section analytically and compare the results with the numerical solution in RFEM 5 and RSTAB 8 for various wall-thickness s respectively for various inner diameters D_1.

Geometry	Tube Cross-section	Outer Diameter	D_2	51.000	mm
		Wall Thickness Range	s	2.600 - 10.000	mm

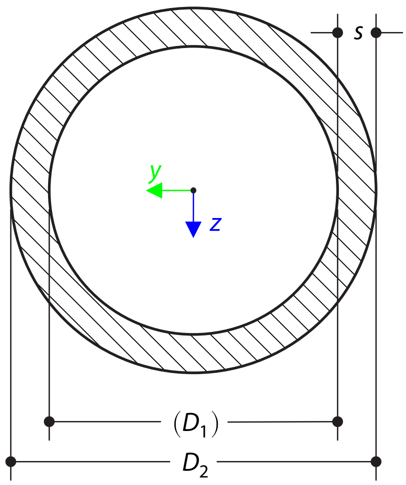

Figure 1: Annular cross-section

Analytical Solution

The torsional constant J is defined as follows:

$$M_x = GJ\vartheta \qquad (1-1)$$

where M_x is the torque, G is the shear modulus and ϑ is the relative rotation of the profile. The torsional constant J can be determined by means of the following process [1], for profiles without holes one has

$$J = 2\int_A \psi(y,z)\mathrm{d}A \qquad (1-2)$$

and for profiles with holes one has

[1] The torsional constant J can be also denoted as I_T.

$$J = 2\int_A \psi(y,z)\mathrm{d}A + 2\sum_i c_i|A_i| \qquad (1-3)$$

where $c_i = \psi(\partial A_i)$ is the constant solution on the inner boundaries, $|A_i|$ is the area of holes and $\psi(y,z)$ is an unknown stress function, which is the solution of the Poisson's partial differential equation of the form

$$\frac{\partial^2\psi(y,z)}{\partial y^2} + \frac{\partial^2\psi(y,z)}{\partial z^2} = -2 \qquad (1-4)$$

on the profile using the boundary condition $\psi(y,z) = c$, where c is the constant. This condition can be further specified, $\psi = 0$ is taken on the outer boundaries and $\psi = c$ on the inner boundaries. For the given annular cross-section the torsional constant J and the polar moment of inertia[2] I_p should coincide and are defined by the known formula

$$J = I_\mathrm{p} = \frac{\pi}{32}\left(D_2^4 - D_1^4\right) \qquad (1-5)$$

This can be proved by solving the formula **(1 – 4)**. Due to the symmetry the stress function, ψ is not function of the polar coordinate φ, i.e. $\psi \neq \psi(\varphi)$. Laplacian operator $\Delta\psi$ has the following form in the polar coordinates

$$\Delta\psi = \psi_{yy} + \psi_{zz} = \psi_{rr} + \frac{\psi_r}{r} + \frac{\psi_{\varphi\varphi}}{r^2} = \psi_{rr} + \frac{\psi_r}{r} \qquad (1-6)$$

The problem is now described by the following equation and boundary condition

$$\Delta\psi = \psi_{rr} + \frac{\psi_r}{r} = -2 \qquad (1-7)$$

$$\psi(R_2) = 0,\ R_2 > R_1 \qquad (1-8)$$

where $R_i = \frac{D_i}{2}$, for $i = 1, 2$ is the tube radius. The following solution is assumed

$$\psi = Ar^2 + B \qquad (1-9)$$

where A and B are unknown constants, which can be obtained from formulae **(1 – 7)** and **(1 – 8)**. The stress function ψ then results

$$\psi = \frac{R_2^2 - r^2}{2} \qquad (1-10)$$

The constant c (solution on the inner boundaries) can be now calculated

[2] The polar moment of inertia is defined as $I_\mathrm{p} = \int_A (y^2+z^2)\mathrm{d}A$ and for given profile can be calculated as follows: $I_\mathrm{p} = 2\pi\int_{R_1}^{R_2} r^3\mathrm{d}r = \frac{\pi}{2}\left(R_2^4 - R_1^4\right) = \frac{\pi}{32}\left(D_2^4 - D_1^4\right)$.

$$c = \psi(R_1) = \frac{R_2^2 - R_1^2}{2} \tag{1 – 11}$$

The torsional constant J can be finally obtained according to the formula **(1 – 3)**

$$J = 4\pi \int_{R_1}^{R_2} \psi(r) r \,\mathrm{d}r + 2\frac{R_2^2 - R_1^2}{2}\pi R_1^2 = \frac{\pi}{2}\left(R_2^4 - R_1^4\right) = \frac{\pi}{32}\left(D_2^4 - D_1^4\right) \tag{1 – 12}$$

This proves the equality of the torsional constant J and the polar moment of inertia I_p for the given annular cross-section. The formula **(1 – 12)** can be rewritten into the form

$$J = \frac{\pi}{4}\left(D_2^3 s - 3D_2^2 s^2 + 4D_2 s^3 - 2s^4\right) \tag{1 – 13}$$

where $s = R_2 - R_1$ is the tube thickness. According to the theory for thin-walled cross-sections the torsional constant J can be calculated as

$$J = 2A_m s \frac{D_2 - s}{2} = \frac{\pi}{4}\left(D_2^3 s - 3D_2^2 s^2 + 3D_2 s^3 - s^4\right) \tag{1 – 14}$$

where A_m is the area limited by the midline of the cross-section. It is obvious, that the formulae differ in the last two terms.

RFEM 5 and RSTAB 8 Settings

- Modeled in RFEM 5.04.0024, RSTAB 8.04.0024 and SHAPE-MASSIVE 6.54

Results

Structure File	Program	Tube Dimension
0001.01	RFEM 5	51 x 2.6
0001.02	RFEM 5	51 x 5
0001.03	RFEM 5	51 x 10
0001.04	RSTAB 8	51 x 2.6
0001.05	RSTAB 8	51 x 5
0001.06	RSTAB 8	51 x 10

Remark: In RFEM 5 / RSTAB 8 is the equation **(1 – 4)** used for all cross-sections except of those, which are manufactured according to the Canadian and U.S. standards. For these cross-sections the torsional constant J taken from these standards is prefered.

Wall thickness	Analytical Solution	RFEM 5 / RSTAB 8 (Rolled Cross-section)		RFEM 5 / RSTAB 8 (Parametric Thin-Walled Cross-section)		RFEM 5 / RSTAB 8 (Parametric Massive Cross-section)	
s [mm]	*J* [mm^4]	*J* [mm^4]	Ratio [-]	*J* [mm^4]	Ratio [-]	*J* [mm^4]	Ratio [-]
2.600	232194	232194	1.000	232194	1.000	232194	1.000
5.000	386754	386754	1.000	386754	1.000	386754	1.000
10.000	573506	573506	1.000	573506	1.000	573506	1.000

Wall thickness	Analytical Solution	SHAPE-MASSIVE	
s [mm]	*J* [mm^4]	*J* [mm^4]	Ratio [-]
2.600	232194	222660	0.959
5.000	386754	373952	0.967
10.000	573506	561682	0.979

References

[1] WUNDERLICH, W. and KIENER, G. *Statik der Stabtragwerke*. Teubner, 2004.