0048 – Uniaxial Bending with Pressure

Description

A structure made of I-profile is fully fixed on the left end ($x = 0$) and embedded into the sliding support on the right end. The structure consists of two segments according to the **Figure 1** [1]. The problem is described by the following set of parameters.

Material	Steel	Modulus of Elasticity	E	210000.000	MPa
		Poisson's Ratio	ν	0.300	–
Geometry	Structure	Segment 1 Length	L_1	6.000	m
		Segment 2 Length	L_2	1.200	m
	Cross-Section	Height	h	400.000	mm
		Width	b	180.000	mm
		Web Thickness	s	10.000	mm
		Flange Thickness	t	14.000	mm
Load		Axial Force	F_x	100.000	kN
		Transverse Force	$F_z = F_x/200$	0.500	kN

The self-weight is neglected in this example. Determine the maximum deflection of the structure $u_{z,\text{max}}$, the bending moment M_y on the fixed end, the rotation $\varphi_{2,y}$ of the segment 2 and the reaction force R_{Bz} by means of the Geometrically linear analysis and the second-order analysis.

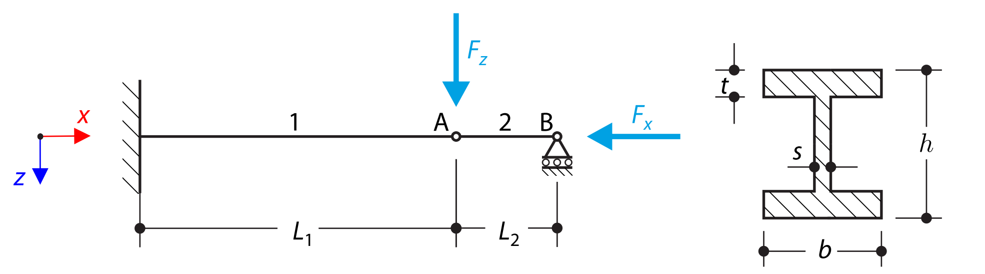

Figure 1: Problem sketch

Analytical Solution

Geometrically linear analysis is carried out at first. In this case, the axial force F_x is not taken into account. The problem can be then solved as well as a cantilever of the length L_1 loaded only by the transverse force F_z. The maximum deflection $u_{z,\max}$ can be calculated using Mohr's integral and results into well-known expression

$$u_{z,\max} = \frac{F_z L_1^3}{3EI_y} = 0.743 \text{ mm} \tag{48 – 1}$$

where I_y is the quadratic moment of the cross-section to the y-axis[1]. The bending moment on the fixed end can be calculated according to the following formula

$$M_y(0) = F_z L_1 = 3.000 \text{ kNm} \tag{48 – 2}$$

The rotation of the segment 2 $\varphi_{2,y}$ is calculated from the geometric condition as follows

$$\varphi_{2,y} = \arctan\left(\frac{u_{z,\max}}{L_2}\right) = 0.619 \text{ mrad} \tag{48 – 3}$$

The reaction force in the sliding joint R_{Bz} can be obtained from the free body diagram shown in the **Figure 2** as

$$R_{Bz} = -\frac{F_x u_{z,\max}}{L_2} = 0.000 \text{ kN} \tag{48 – 4}$$

considering the zero effect of the axial force F_x. Because of the nonnegligible effect of the axial force F_x the second-order analysis should be considered. Thus the axial force F_x is taken into account and produces another contribution to the bending moment. The problem can be described by the free body diagram of the segments according to the **Figure 2**.

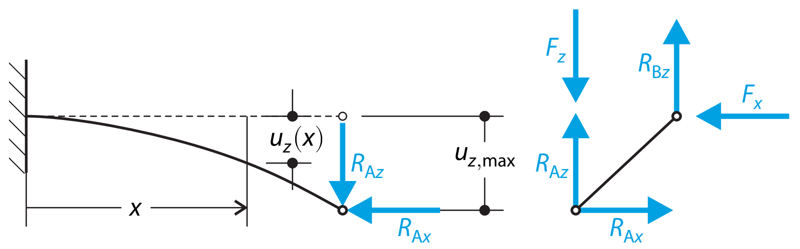

Figure 2: Free body of the structure

The unknown reaction forces can be obtained from the equilibrium equations.

$$x: \quad R_{Ax} = F_x \tag{48 – 5}$$
$$y: \quad R_{Az} + R_{Bz} - F_z = 0 \tag{48 – 6}$$
$$M_{yA}: \quad F_x u_{z,\max} + R_{Bz} L_2 = 0 \tag{48 – 7}$$

The segment 1 is obviously loaded by the reaction forces R_{Ax} and R_{Az}

[1] $I_y = \frac{1}{12}t(h-2s)^3 + \frac{1}{6}bs^3 + \frac{sb}{2}(h-s)^2 = 2.307 \cdot 10^8 \text{ mm}^4$

$$R_{\mathrm{A}x} = F_x \quad (48-8)$$

$$R_{\mathrm{A}z} = F_z + \frac{F_x u_{z,\mathrm{max}}}{L_2} \quad (48-9)$$

which causes the total bending moment M_y

$$M_y = -R_{\mathrm{A}z}\left(L_1 - x\right) - R_{\mathrm{A}x}\left(u_{z,\mathrm{max}} - u_z(x)\right) \quad (48-10)$$

where $u_{z,\mathrm{max}}$ is the deflection at the point $x = L_1$. The solution can be found by the Euler-Bernoulli differential equation

$$\frac{\mathrm{d}^2 u_z}{\mathrm{d}x^2} = -\frac{M_y}{EI_y} \quad (48-11)$$

It can be rewritten into the form

$$\frac{\mathrm{d}^2 u_z}{\mathrm{d}x^2} + \alpha^2 u_z = -\frac{1}{EI_y}\left(F_z + \frac{F_x u_{z,\mathrm{max}}}{L_2}\right)x + \frac{1}{EI_y}\left(F_z L_1 + \frac{F_x u_{z,\mathrm{max}} L_1}{L_2} + F_x u_{z,\mathrm{max}}\right) \quad (48-12)$$

where α is defined as

$$\alpha = \sqrt{\frac{F_x}{EI_y}} \quad (48-13)$$

The total solution consists of the homogeneous and the particular solution

$$u_z = C_1 \cos(\alpha x) + C_2 \sin(\alpha x) + u_{z\mathrm{P}} \quad (48-14)$$

where C_1 and C_2 are the unknown constants, which can be obtained from the boundary conditions. The particular solution $u_{z\mathrm{P}}$ can be found in the form of the linear function

$$u_{z\mathrm{P}} = C_3 x + C_4 \quad (48-15)$$

where constants C_3 and C_4 can be calculated by substituting the particular solution and its derivatives into the differential equation **(48 – 12)**. The constants then results

$$C_3 = -\frac{F_z}{F_x} - \frac{u_{z,\mathrm{max}}}{L_2} \quad (48-16)$$

$$C_4 = \frac{F_z L_1}{F_x} + \frac{u_{z,\mathrm{max}} L_1}{L_2} + u_{z,\mathrm{max}} \quad (48-17)$$

The boundary conditions are obvious from the **Figure 2**.

$$u_z(0) = 0 \tag{48 – 18}$$

$$u_z'(0) = 0 \tag{48 – 19}$$

$$u_z(L_1) = u_{z,\max} \tag{48 – 20}$$

From conditions **(48 – 18)** and **(48 – 19)** results constants C_1, C_2.

$$C_1 = -C_4 \tag{48 – 21}$$

$$C_2 = -\frac{C_3}{\alpha} \tag{48 – 22}$$

The constant $u_{z,\max}$, which is the desired solution, results from the condition **(48 – 20)**

$$u_{z,\max} = \frac{F_z L_2 \left[\alpha L_1 \cos(\alpha L_1) - \sin(\alpha L_1)\right]}{F_x \left[\alpha \cos(\alpha L_1)(L_1 + L_2) - \sin(\alpha L_1)\right]} = 0.878 \text{ mm} \tag{48 – 23}$$

The bending moment on the fixed end can be calculated according to the following formula

$$M_y(0) = R_{\text{Az}} L_1 + R_{\text{Ax}} u_{z,\max} = 3.527 \text{ kNm} \tag{48 – 24}$$

The rotation of the segment 2 $\varphi_{2,y}$ is calculated from the geometric condition as follows

$$\varphi_{2,y} = \arctan\left(\frac{u_{z,\max}}{L_2}\right) = 0.732 \text{ mrad} \tag{48 – 25}$$

The reaction force in the sliding joint R_{Bz} results

$$R_{\text{Bz}} = -\frac{F_x u_{z,\max}}{L_2} = -0.073 \text{ kN} \tag{48 – 26}$$

The general solution of the deflection $u_z(x)$ valid in the interval $x \in [0, L_1]$ can be written as follows

$$u_z(x) = \frac{F_z L_2 \left[-\cos(\alpha L_1)\alpha x + \cos(\alpha L_1)\sin(\alpha x) - \sin(\alpha L_1)\cos(\alpha x) + \sin(\alpha L_1)\right]}{F_x \left[\alpha L_1 \cos(\alpha L_1) + \alpha L_2 \cos(\alpha L_1) - \sin(\alpha L_1)\right]} \tag{48 – 27}$$

It is obvious that the influence of the axial force F_x is considerable. The total deflection of the structure under the prescribed loading in case of the second-order analysis is approximately 18 % greater than in case of geometrically linear analysis. The comparison of the Geometrically linear analysis and the second-order analysis is shown in the **Figure 3**, considering the ratio of the loading forces $F_z = F_x/200$. It is obvious that the difference between these analysis is more considerable when the loading is grater. The second-order analysis solution is approaching the horizontal asymptote. The position of this asymptote can be calculated from the equation **(48 – 23)** for $u_{z,\max}$ approaching the infinity, which means that the denominator equals zero.

$$\tan(\alpha L_1) - \alpha(L_1 + L_2) = 0 \qquad (48 – 28)$$

From the numerical solution of the equation **(48 – 28)** results the value of the horizontal asymptote $F_{x,\mathrm{cr}} = 650.873$ kN.

RFEM and RSTAB Settings

- Modeled in RFEM 5.05.0029 and RSTAB 8.05.0029 and RFEM 6.01, RSTAB 9.01
- The number of elements is 2 (one element per member)
- The number of increments is 5
- Isotropic linear elastic material model is used
- The structure is modeled using members
- Shear stiffness of the members is neglected

Results

Structure Files	Program	Method of Analysis
0048.01	RSTAB 8, RSTAB 9	Geometrically Linear Analysis
0048.02	RSTAB 8, RSTAB 9	Second-Order Analysis
0048.03	RFEM 5, RFEM 6	Geometrically Linear Analysis
0048.04	RFEM 5, RFEM 6	Second-Order Analysis

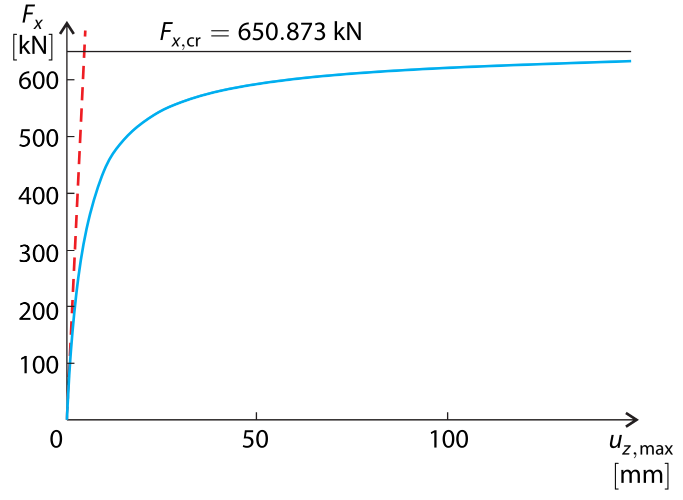

Figure 3: The comparison of the Geometrically linear analysis (dashed line) and the second-order analysis (solid line).

Method of Analysis	Analytical Solution	RSTAB 8		RFEM 5	
	$u_{z,\max}$ [mm]	$u_{z,\max}$ [mm]	Ratio [-]	$u_{z,\max}$ [mm]	Ratio [-]
Geometrically Linear Analysis	0.743	0.743	1.000	0.743	1.000
Second-Order Analysis	0.878	0.878	1.000	0.878	1.000

Method of Analysis	Analytical Solution	RSTAB 9		RFEM 6	
	$u_{z,\max}$ [mm]	$u_{z,\max}$ [mm]	Ratio [-]	$u_{z,\max}$ [mm]	Ratio [-]
Geometrically Linear Analysis	0.743	0.743	1.000	0.743	1.000
Second-Order Analysis	0.878	0.878	1.000	0.878	1.000

Method of Analysis	Analytical Solution	RSTAB 8		RFEM 5	
	$M_y(0)$ [kNm]	$M_y(0)$ [kNm]	Ratio [-]	$M_y(0)$ [kNm]	Ratio [-]
Geometrically Linear Analysis	3.000	3.000	1.000	3.000	1.000
Second-Order Analysis	3.527	3.527	1.000	3.527	1.000

Method of Analysis	Analytical Solution	RSTAB 9		RFEM 6	
	$M_y(0)$ [kNm]	$M_y(0)$ [kNm]	Ratio [-]	$M_y(0)$ [kNm]	Ratio [-]
Geometrically Linear Analysis	3.000	3.000	1.000	3.000	1.000
Second-Order Analysis	3.527	3.527	1.000	3.527	1.000

Method of Analysis	Analytical Solution	RSTAB 8		RFEM 5	
	$\varphi_{2,y}$ [mrad]	$\varphi_{2,y}$ [mrad]	Ratio [-]	$\varphi_{2,y}$ [mrad]	Ratio [-]
Geometrically Linear Analysis	0.619	0.619	1.000	0.619	1.000
Second-Order Analysis	0.732	0.732	1.000	0.732	1.000

Method of Analysis	Analytical Solution	RSTAB 9		RFEM 6	
	$\varphi_{2,y}$ [mrad]	$\varphi_{2,y}$ [mrad]	Ratio [-]	$\varphi_{2,y}$ [mrad]	Ratio [-]
Geometrically Linear Analysis	0.619	0.619	1.000	0.619	1.000
Second-Order Analysis	0.732	0.732	1.000	0.732	1.000

Method of Analysis	Analytical Solution	RSTAB 8		RFEM 5	
	R_{Bz} [kN]	R_{Bz} [kN]	Ratio [-]	R_{Bz} [kN]	Ratio [-]
Geometrically Linear Analysis	0.000	0.000	-	0.000	-
Second-Order Analysis	-0.073	-0.073	1.000	-0.073	1.000

Method of Analysis	Analytical Solution	RSTAB 9		RFEM 6	
	R_{Bz} [kN]	R_{Bz} [kN]	Ratio [-]	R_{Bz} [kN]	Ratio [-]
Geometrically Linear Analysis	0.000	0.000	-	0.000	-
Second-Order Analysis	-0.073	-0.073	1.000	-0.073	1.000

References

[1] LUMPE, G. and GENSICHEN, V. *Evaluierung der linearen und nichtlinearen Stabstatik in Theorie und Software: Prüfbeispiele, Fehlerursachen, genaue Theorie*. Ernst, 2014.