Program: RFEM 5, RF-DYNAM Pro

Category: Isotropic Linear Elasticity, Dynamics, Plate

Verification Example: 0108 – Natural Vibrations of Circular Membrane

0108 – Natural Vibrations of Circular Membrane

Description

A circular membrane is tensioned by a line force N according to **Figure 1**. Determine the natural frequencies of the circular membrane. The problem is described by the following parameters.

Material	Steel	Modulus of Elasticity	E	210000.0	MPa
		Poisson's Ratio	ν	0.296	–
		Density	ρ	7850.000	kgm^{-3}
Geometry		Radius	a	0.500	m
		Thickness	h	0.001	m
Load		Line Force	N	100.000	kN/m

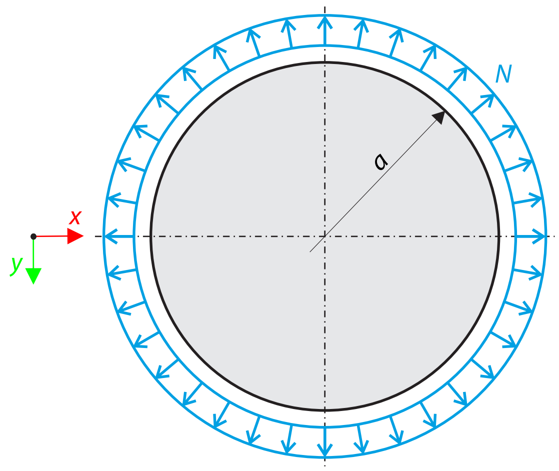

Figure 1: Problem Sketch

Analytical Solution

Free vibrations of a circular membrane can be described by the wave equation in polar coordinates, where the deflection $u_z(r, \varphi, t)$ is a function of the radius r, the angle φ and time t, namely

$$\frac{\partial^2 u_z}{\partial r^2} + \frac{1}{r}\frac{\partial u_z}{\partial r} + \frac{\partial^2 u_z}{\partial \varphi^2} - \frac{1}{c^2}\frac{\partial^2 u_z}{\partial t^2} = 0 \qquad (108-1)$$

The speed of the wave propagation c is given by the density ρ and thickness h of the membrane, and the tension force N

$$c = \sqrt{\frac{N}{\rho h}} \qquad (108-2)$$

The solution is sought for through separation of variables

$$u_z(r,\varphi,t) = R(r)F(\varphi)T(t) \tag{108 – 3}$$

Hence, substituting into **(108 – 1)** yields the following form[1]

$$\frac{\ddot{T}}{T} = c^2\left(\frac{R''}{R} + \frac{1}{r}\frac{R'}{R} + \frac{F''}{F}\right) = -\Omega^2 \tag{108 – 4}$$

The left-hand side depends on time t, while the right-hand side only on the spatial coordinates r and φ. Thus both sides have to be equal to a constant, which can be shown to be negative, $-\Omega^2$, for some $\Omega > 0$.

The first part of **(108 – 4)**

$$\ddot{T} + \Omega^2 T = 0 \tag{108 – 5}$$

yields a solution in the following form

$$T(t) = A\sin(\Omega t) + B\cos(\Omega t) \tag{108 – 6}$$

where the constants A, B depend on the initial conditions.

It follows from the second part of **(108 – 4)** that both sides have to be equal to a constant m^2, as the two terms on the left-hand side contain variables that are independent from each other, more precisely,

$$r^2\frac{R''}{R} + r\frac{R'}{R} + r^2\frac{\Omega^2}{c^2} = m^2 = -\frac{F''}{F}, \qquad m = 0, 1, 2, \dots \tag{108 – 7}$$

The solution of the right-hand-side equation

$$F'' + m^2 F = 0 \tag{108 – 8}$$

is, analogously to **(108 – 5)**,

$$F(\varphi) = C\sin(m\varphi) + D\cos(m\varphi) \tag{108 – 9}$$

where the constants C, D depend on the boundary conditions.

The left-hand-side equation of **(108 – 7)** can be adjusted into the Bessel differential equation

[1] The dashed notation indicates the derivative with respect to appropriate coordinate, e.g. $R'' = \frac{\mathrm{d}^2R(r)}{\mathrm{d}r^2}$. The dotted notation indicates the derivative with respect to time t.

$$r^2R'' + rR' + \left(r^2\frac{\Omega^2}{c^2} - m^2\right)R = 0 \qquad (108-10)$$

the solution of which reads as

$$R(r) = EJ_m\left(\frac{\Omega_{mn}}{c}r\right) + FY_m\left(\frac{\Omega_{mn}}{c}r\right) \qquad (108-11)$$

Again, the constants E, F depend on the boundary conditions. J_m and Y_m are the m-th order Bessel functions of the first and second kind, respectively. The Bessel function Y_m is unbounded for $r \longrightarrow 0$, which results in an unphysical solution to the vibrating membrane problem, thus the constant F must be zero.

The boundary condition for the zero membrane deflection on the circumference can be written as

$$R(a) = 0 \qquad (108-12)$$

The problem of natural vibrations of the circular membrane is then described by **(108 – 13)**, where the lower index n denotes the number of the Bessel function root (Bessel functions of the first kind and various order J_m are shown in **Figure 2**)

$$J_m\left(\frac{\Omega_{mn}}{c}a\right) = 0, \qquad n = 1, 2, 3, \ldots \qquad (108-13)$$

Considering that $\Omega_{mn} = 2\pi f_{mn}$, the natural frequencies of the rectangular membrane can be calculated as the roots of the Bessel functions, more precisely

$$J_m\left(\frac{2\pi a}{c}f_{mn}\right) = 0 \qquad (108-14)$$

For clarity, **Figure 2** shows some Bessel functions J_m of the first kind and various order.

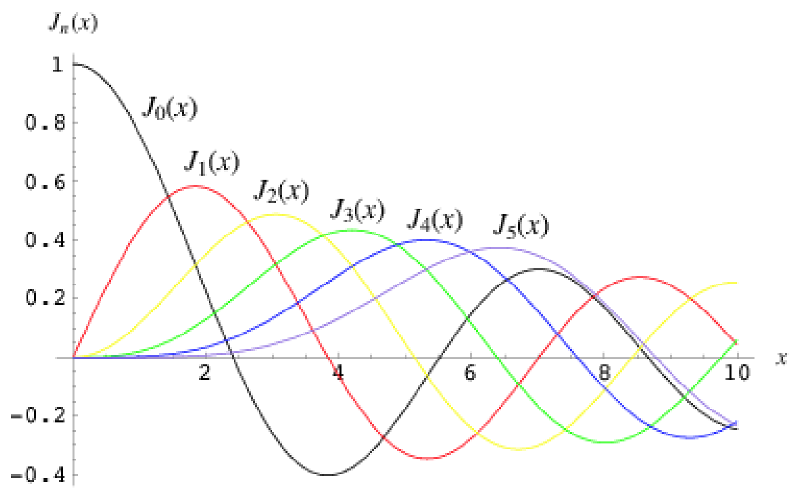

Figure 2: Bessel functions J_m of the first kind and various order, cf. [1]

RFEM 5 Settings

- Modeled in RFEM 5.16.01
- The element size is $l_{FE} = 0.020$ m
- Isotropic linear elastic material model is used

Results

Structure Files	Program
0108.01	RFEM 5 – RF-DYNAM Pro

Results

Quantity	Analytical Solution	RF-DYNAM Pro	Ratio
f_1 [Hz]	86.397	86.346	0.999
f_2 [Hz]	137.660	137.627	1.000
f_3 [Hz]	184.505	184.261	0.999
f_4 [Hz]	198.317	197.856	0.998
f_5 [Hz]	229.217	228.848	0.998
f_6 [Hz]	252.046	251.146	0.996

Following **Figure 3** shows the first six natural shapes of the investigated membrane.

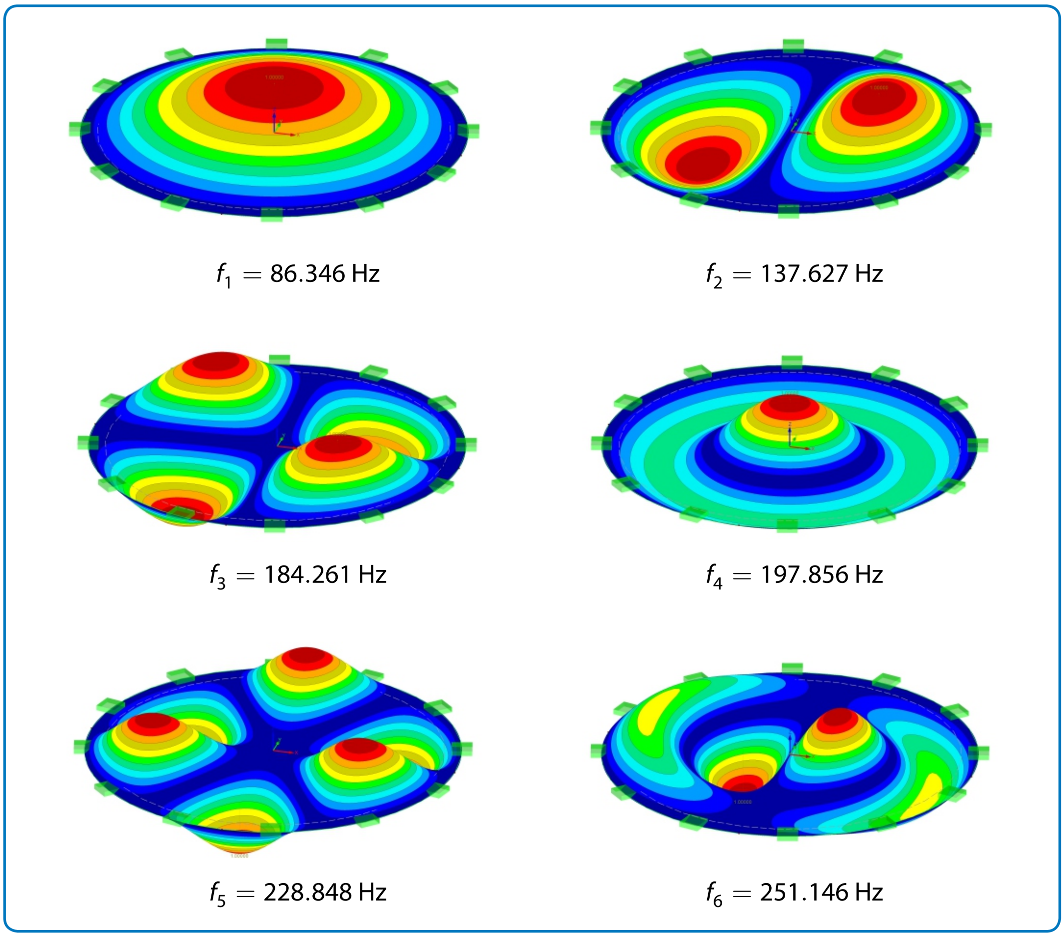

Figure 3: First six natural shapes of the circular membrane in RFEM 5

References

[1] http://mathworld.wolfram.com/BesselFunctionoftheFirstKind.html, *Bessel function of the first kind*